Ref.: Henrici: "Applied & Computational Complex Analysis" (Wiley)

Formal manipulation of power series: $A = \sum_0^\infty a_n z^n$, $B = \sum_0^\infty b_n z^n$

Product: $AB = C$

$$\left(\sum a_n z^n\right)\left(\sum b_n z^n\right) = \sum c_n z^n$$

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$ (Cauchy product formula)

Matrix representation:

$$A \longrightarrow P = \begin{pmatrix} a_0 & a_1 & a_2 & \ddots \\ & a_0 & a_1 & \ddots \\ & & a_0 & \ddots \\ 0 & & & \ddots \end{pmatrix}$$

("semicirculant" matrix)

$A \to P$, $B \to Q$

$AB \to PQ$ (check this!)

Since upper triangular: $(AB)_n = A_n B_n$

$(A)_n$: upper $n \times n$ block

(will use same letter for series & matrix if clear in context)

Inversion: if A has $a_0 \neq 0$

$$A^{-1} = \frac{1}{A} = \frac{1}{a_0 + a_1 z + a_2 z^2 + \cdots} : 1 = AA^{-1} = A^{-1}A$$

let $A^{-1} = b_0 + b_1 z + b_2 z^2 + \cdots$

If $A^{-1} \to S$, $SP = I$

$$a_0 b_0 = 1$$

$$\begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix} \begin{pmatrix} b_0 & b_1 \\ 0 & b_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \quad a_0 b_1 + a_1 b_0 = 0$$

Similarly: $a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \Rightarrow$

$$\cdots \quad b_n = -(a_1 b_{n-1} + \cdots + a_n b_0)/a_0$$

Wronski formulas: $b_n = \dfrac{(-1)^n}{a_0^{n+1}} \det \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_0 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & a_2 \\ 0 & & a_0 & a_1 \end{pmatrix}$

i.e. $b_0 = 1/a_0$, $b_1 = -a_1/a_0^2$, $b_2 = \dfrac{1}{a_0^3} \begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}$, $b_3 = -\dfrac{1}{a_0^4} \begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}$

Ex: $a_0 = 1,\ a_1 = 1,\ a_{n+1} = a_n + a_{n-1}$

$(A = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + \cdots)$

Find A^{-1}: $b_0 = 1$, $b_1 = -\frac{a_1}{a_0^2} = -1$

$$b_2 = \frac{1}{1}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$$

$$b_3 = -\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$$b_k = \frac{(-1)^k}{a_0^{k+1}}\begin{vmatrix} 1 & 2 & 3 & 5 & 8 & \cdots \\ 1 & 1 & 2 & 3 & 5 & \\ 0 & 1 & 1 & 2 & 3 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{vmatrix} = \frac{(-1)^k}{a_0^{k+1}}\begin{vmatrix} 1 & 2 & 3 & 5 & 8 & \cdots \\ 1 & 2 & 3 & 5 & 8 & \cdots \\ 0 & 1 & 1 & 2 & 3 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{vmatrix} = 0$$

i.e. $\frac{1}{A} = 1 - z - z^2$ (adding rows 2+3 gives row 1, by virtue of recurrence relation)

(Similar results for all series whose coefficients satisfy simple recurrence relations)

Formal differentiation and D.E.

$$\frac{dA}{dz} = a_1 + 2a_2 z + \cdots + n a_n z^{n} + \cdots$$

Ex: $E_\alpha(x) = 1 + \frac{\alpha x}{1!} + \cdots + \frac{\alpha^n x^n}{n!} + \cdots$ exponential series

satisfies $P' = \alpha P$

$$(a_1 + 2a_2 z + \cdots) = \alpha(a_0 + a_1 z + \cdots)$$

Equating powers of z: $n a_n = \alpha a_{n-1} \Rightarrow a_n = \frac{\alpha^n}{n!} a_0$

$\Rightarrow P = a_0 E_\alpha(z)$; show $E_a E_b = E_{a+b}$

// let $P = E_a E_b - E_{a+b}$

$P' = a E_a E_b + b E_a E_b - (a+b) E_{a+b} = (a+b) P$

$\Rightarrow P = c_0 E_{a+b}$; but $c_0 = 0$ QED

$(1 + az + \cdots)(1 + bz + \cdots) - (1 + (a+b)z + \cdots) = 0 + 0z \cdots$

Alternatively:

$$E_a E_b = \sum_0^\infty c_n z^n$$

Cauchy: $c_n = \sum_{k=0}^{n}{}' \dfrac{a^k b^{n-k}}{k!(n-k)!} = \dfrac{(a+b)^n}{n!}$

$$\Rightarrow (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$ binomial formula

<u>Example</u>: $P' = \dfrac{\alpha}{1+z} P$ $\quad (P = (1+z)^\alpha$; α arbitrary).

$$(1+z)(b_1 + 2b_2 z + \dots) = \alpha(b_0 + b_1 z + b_2 z^2 + \dots)$$

$$b_1 + (2b_2 + b_1) z + (3b_3 + 2b_2) z^2 + \dots = \alpha b_0 + \alpha b_1 z + \alpha b_2 z^2 + \dots$$

$$\Rightarrow n b_n + (n-1) b_{n-1} = \alpha b_{n-1} \quad \Rightarrow b_n = \frac{\alpha - n + 1}{n} b_{n-1} = \binom{\alpha}{n} b_0$$

with: $\binom{\alpha}{0} := 1$, $\binom{\alpha}{1} = \alpha$, $\binom{\alpha}{n} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$

$$P = b_0 Q_\alpha(z) \quad ; \quad Q_\alpha(z) = 1 + \binom{\alpha}{1} z + \binom{\alpha}{2} z^2 + \dots$$

Pochhammer symbol

$$(\alpha)_n := \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$$

Then $\binom{\alpha}{n} = (-1)^n \frac{(-\alpha)_n}{n!}$

Hypergeometric series:

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n\, n!} z^n$$

Examples: ${}_1F_0(1;z) = 1+z+z^2+\cdots$ geometric series

$E_\alpha(z) = {}_0F_0(\alpha z) = 1 + \frac{\alpha z}{1!} + \frac{\alpha^2 z^2}{2!} + \cdots$ Exponential

$Q_\alpha(z) = 1 + \binom{\alpha}{1} z + \binom{\alpha}{2} z^2 + \cdots = {}_1F_0(-\alpha;-z)$ Binomial

$J_0(z) = 1 - \frac{z^2}{2^2(1!)^2} + \frac{z^4}{2^4(2!)^2} - \frac{z^6}{2^6(3!)^2} + \cdots = {}_0F_1(1;-\frac{z^2}{4})$ Bessel

Inverses when $a_0 = 0$. (Formal Laurent series)

$$A = z^m (b_0 + b_1 z + \cdots) = z^m B, \quad b_0 \neq 0$$

$$A^{-1} = z^{-m} B^{-1}$$; find B^{-1} as before

i.e.

$$A^{-1} = \frac{a_{-m}}{z^m} + \frac{a_{-m+1}}{z^{m-1}} + \cdots + \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots$$

For this series to represent a function of z (i.e. converge) we need $z \neq 0$: i.e. such a series will be valid in the punctured disk $0 < |z| < R$ where R is the radius of convergence of $B^{-1}(z)$.

Such series are called <u>Laurent</u> series (in general they are valid in annuli $a < |z - z_0| < b$)

Example $f(z) = \dfrac{1}{z-1}$, $|z|>1$:

$$f = \frac{1}{z}\left(\frac{1}{1-1/z}\right) \quad ; \quad |1/z| < 1, \text{ use geometric}$$

$$= \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

Example: $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$, $1 < |z| < 2$

$$\frac{1}{z-2} = -\frac{1}{2}\,\frac{1}{1-(z/2)} = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \cdots\right), \quad |z| < 2$$

$$\frac{1}{z-1} = \frac{1}{z}\left(\frac{1}{1-1/z}\right) = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right), \quad |z| > 1$$

$$f(z) = -\sum_{n=1}^{\infty} \frac{1}{z^n} - \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n = \sum_{n=-\infty}^{\infty}{}' \, b_n z^n, \quad 1 < |z| < 2$$

Laurent Expansion: let $f(z)$ single valued, analytic in $0<r_2<|z-z_0|<r_1$

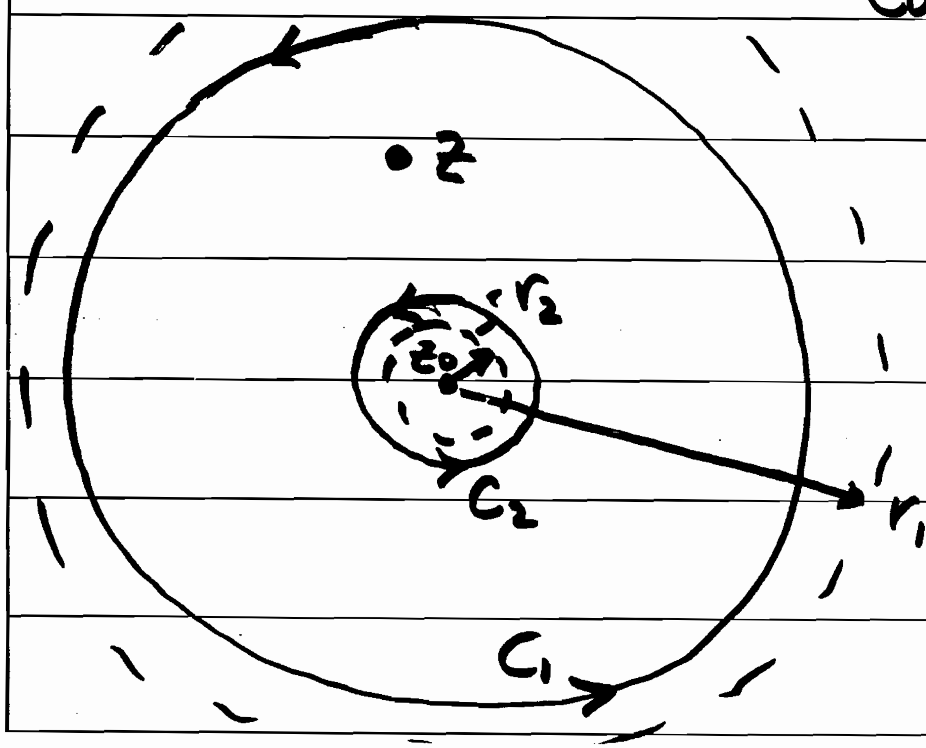
Obtain expansion convergent for

$\to$ convergent

$$r_2 < \underbrace{\rho_2 < |z-z_0| < \rho_1}_{\text{uniformly convergent}} < r_1$$

Cauchy:

$$f(z)=\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta-z}\,d\zeta - \frac{1}{2\pi i}\oint_{C_2}\frac{f(\zeta)}{\zeta-z}\,d\zeta$$



$$C_1: \quad \frac{1}{\zeta - z} = \cdots = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}} \quad \left(\begin{matrix}\text{like} \\ \text{Taylor}\end{matrix}\right)$$

uniformly conv. on C_1 since $\left|\frac{z-z_0}{\zeta - z_0}\right| < \frac{\rho}{\rho_1} < 1$

$$C_2: \quad \frac{1}{\zeta - z} = -\frac{1}{z-z_0}\,\frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} = -\sum_{n=0}^{\infty} \frac{(\zeta - z_0)^n}{(z-z_0)^{n+1}}$$

u.c. on C_2 since $\left|\frac{\zeta - z_0}{z - z_0}\right| = \frac{\rho_2}{\rho} < 1$

$$\therefore f(z) = \sum_{n=0}^{\infty} \left\{ \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right\} (z-z_0)^n$$

$$+ \sum_{n=0}^{\infty} \left\{ \frac{1}{2\pi i} \oint_{C_2} f(\zeta)(\zeta - z_0)^n d\zeta \right\} (z - z_0)^{-(n+1)}$$

$$= \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad ; \qquad j = \begin{cases} 1, & n \geq 0 \\ 2, & n < 0 \end{cases}$$

$$a_n = \frac{1}{2\pi i} \oint_{C_j} f(\zeta)(\zeta - z_0)^{n-1} d\zeta \quad , \quad n = 0, \pm 1, \pm 2, \ldots$$

Residue: let

$$f(z) = \frac{b_{-k}}{z^k} + \dots + \frac{b_{-1}}{z} + b_0 + b_1 z + \dots$$

$$\operatorname{Res}(f) = b_{-1}$$

$f(z)$ has an antiderivative if $\operatorname{Res}(f) = 0$

Indeed: let $F(z) = \frac{a_{-m}}{z^m} + \dots + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \dots$

$(0 < |z| < R)$

$$F'(z) = -m a_{-m} z^{-m-1} + \dots - a_{-1} z^{-2} + 0 + a_1 + 2a_2 z$$

$\hookrightarrow$ **no z^{-1} term!**

To understand Laurent series, consider

$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$; where is it valid?

let $\quad f = f_1(z) + f_2(z)$

$f_1(z) = \sum_{0}^{\infty} a_n (z-z_0)$; converges for

$0 \leq |z-z_0| < \rho_1$ (outer radius of convergence)

$f_2(z) = \sum_{n=-\infty}^{-1} a_n (z-z_0) = \sum_{n=1}^{\infty} a_{-n} \zeta^n$,

$\zeta = \frac{1}{z-z_0}$

converges for $0 \leq |\zeta| < R = 1/\rho_2$

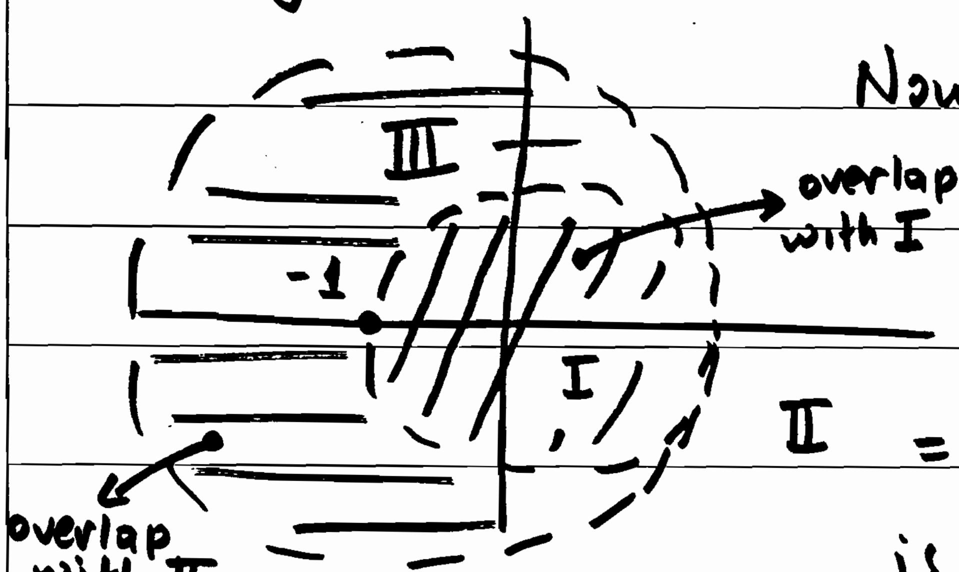
i.e. $\rho_2 < |z-z_0| < \infty$

Combining, we get that f converges in $\rho_1 < |z-z_0| < \rho_2$ where both f_1, f_2 converge (overlap region).

Analytic continuation

$$f(z) = \frac{1}{1-z} = \begin{cases} 1 + z + z^2 + \cdots & |z|<1 \quad \text{(I)} \\ -\left(\frac{1}{z} + \frac{1}{z^2} + \cdots\right), & |z|>1 \quad \text{(II)} \end{cases}$$

Two series have nothing in common (valid at different regions), yet they represent the same function.



Now $f(z) = \dfrac{1}{1-z} = \dfrac{1}{2-(z+1)}$

$$= \frac{1}{2}\,\frac{1}{1-\frac{z+1}{2}} =$$

$$= \frac{1}{2}\left(1 + \frac{z+1}{2} + \left(\frac{z+1}{2}\right)^2 + \cdots\right), \quad |z+1|<2 \quad \text{(III)}$$

is another representation