Math. 504
Set I

1.3, 2.1, 2.5, 2.6, 3.2, 3.3, 3.5

2.5 $S \in \mathbb{C}^{m\times n}$, $S^* = -S$

(a) $\mathcal{L}\{S\} \subset \Im m$ (the spectrum of S is pure imaginary)

◭ From problem (2.3), the eigenvalues of a Hermitian matrix are real; but $(iS)^* = -iS^* = -i(-S) = iS$, so iS is Hermitian; since $\lambda \in \mathcal{L}\{S\} \Leftrightarrow i\lambda \in \mathcal{L}\{iS\}$, ~~iff~~ the spectrum of S is pure imaginary. ▷

(b) "$I - S$ is nonsingular"

◭ If $\lambda \in \mathcal{L}\{S\}$ then $1-\lambda \in \mathcal{L}\{I-S\}$; since $\lambda = iy$ (y real), then $1-\lambda = 1 - iy \neq 0$
$\Rightarrow I - S$ has no zero eigenvalues $\Rightarrow$ nonsingular ▷

(c) $Q = (I-S)^{-1}(I+S)$ unitary

◭ Since $(I-S)$ and $(I+S)$ commute, each commutes with the other's inverse. So:

$$
\begin{aligned}
QQ^* &= (I-S)^{-1}(I+S)\left[(I-S)^{-1}(I+S)\right]^* \\
&= (I-S)^{-1}(I+S)(I+S)^*\left[(I-S)^*\right]^{-1} \\
&= (I-S)^{-1}(I+S)(I+S^*)(I-S^*)^{-1} \\
&= (I-S)^{-1}\left[(I+S)(I-S)\right](I+S)^{-1} = \\
&= (I-S)^{-1}\left[(I-S)(I+S)\right](I+S)^{-1} = I
\end{aligned}
$$
▷

3.5 Is $\|E\|_F = \|u\|_F\|v\|_F$ if $E = uv^H$? (for vectors, F norm and 2-norm are the same)

◭ In general, $\|E\|_F \le \|u\|_F\|v\|_F = \|u\|_2\|v\|_2$

But $\|E\|_F^2 = \sum_{i,j} u_i^2 v_j^2 = \sum_j (u_1^2+\dots+u_n^2)v_j^2 = (u_1^2+\dots+u_n^2)(v_1^2+\dots+v_n^2) = \|u\|_2^2\|v\|_2^2$ (only true if $\operatorname{rank} E = 1$) ▷

2.6 (A little more generally):

Let $E(u,v;\sigma) = I + \sigma u v^H$; show $E^{-1}(u,v;\sigma) = E(u,v;\tau)$

where $\frac{1}{\sigma} + \frac{1}{\tau} = -v^H u$

Indeed: $(I + \sigma u v^H)(I + \tau u v^H) = I + (\sigma + \tau) u v^H + \sigma\tau\, u (v^H u) v^H$

$= I$

$\Rightarrow \sigma + \tau + \sigma\tau v^H u = 0 \Rightarrow \frac{1}{\sigma} + \frac{1}{\tau} = -\frac{1}{v^H u}$

$\therefore (I + u v^H)^{-1} = I + \eta u v^H$; $\eta = \frac{-1}{1 + v^H u}$

This works, provided $v^H u \neq -1$

$\therefore$ if $v^H u = -1$, let $(I + u v^H) x = 0$

$\Rightarrow x = -u (v^H x) \Rightarrow x = \sigma u$, for some scalar σ

$\Rightarrow \sigma u = -u (v^H \cdot \sigma u) = -u \sigma \underbrace{(v^H u)}_{=-1}$

$\Rightarrow \sigma u = \sigma u$, true for any σ; i.e.

$\mathcal{N}(A) = \text{span}\{u\}$

Note: above is special case of the Sherman-Morrison-Woodbury formula:

$$(A + UV^H)^{-1} = A^{-1} - [A^{-1} U (I + V^H A^{-1} U)^{-1} V^H A^{-1}]$$

where $A \in \mathbb{C}^{n \times n}$, $U, V \in \mathbb{C}^{n \times p}$ with rank p.

⟨2.1⟩ A triangular + unitary $\Rightarrow$ A diagonal

Let A : $a_{ij} = 0$, $i > j$ (triangular)

Since A unitary $\|Ax\| = \|x\| \ \forall x$, and the same is true for A^* : $\|A^* x\| = \|x\|$.

Now take $x = \underline{e}_1 \Rightarrow Ax = \underline{a}_1 = \begin{pmatrix} a_{11} \\ 0 \\ | \\ 0 \end{pmatrix}$ while

$A^* x = \begin{pmatrix} \bar{a}_{11} \\ \bar{a}_{12} \\ \vdots \\ \bar{a}_{1n} \end{pmatrix} = \underline{a}_1^*$; Since $\|Ax\| = \|A^* x\| = \|x\| = 1$

we have that $\|\underline{a}_1\|^2 = \|\underline{a}_1^*\|^2 \Rightarrow$

$\Rightarrow |a_{11}|^2 = |a_{11}|^2 + |a_{12}|^2 + \dots + |a_{1n}|^2 \Rightarrow$

$\sum_{j=2}^{n} |a_{1j}|^2 = 0 \Rightarrow a_{1j} = 0,\ j = 2, \dots, n.$

Similarly prove that all off diagonal terms are zero.
(Same proof used to show: {A triangular + normal} $\Rightarrow$ {A diagonal}

⟨2.5⟩ $S \in \mathbb{C}^{m \times n}$, $S^* = -S$.

(a) Let $Sx = \lambda x \Rightarrow x^* S^* = \lambda^* x^*$

Then $x^* S x = \lambda \|x\|^2$ or $-x^* S x = \lambda^* \|x\|^2$

Adding $(\lambda + \lambda^*) \|x\|^2 = 0 \Rightarrow \lambda = -\lambda^*$ or ~~$\lambda \in \mathbb{R}$~~

$\operatorname{Re} \lambda = 0$.

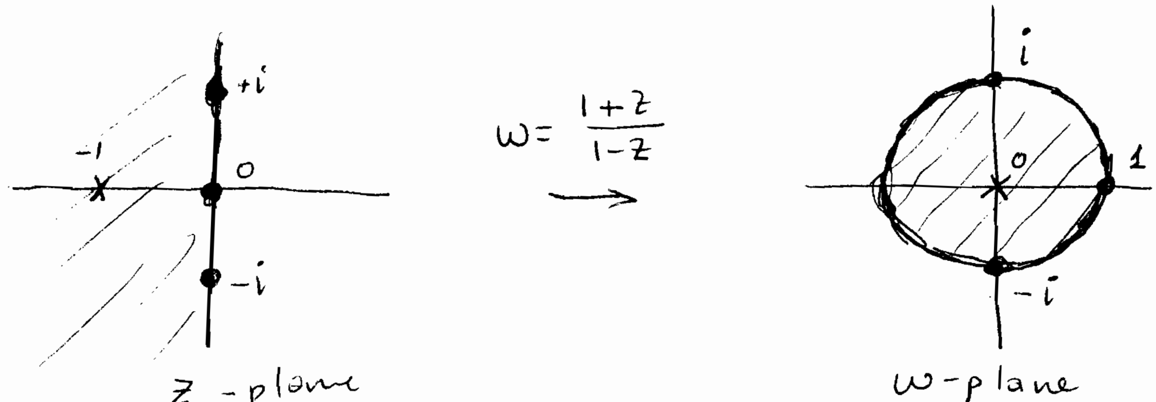
(b) Since 1 is not an eigenvalue of $S \Rightarrow I - S$ nonsingular

(c) Let $U = (I-S)^{-1}(I+S)$; then

$U^* = (I + S^*)(I - S^*)^{-1} = (I-S)(I+S)^{-1}$

4/6 But $(I+S)$ commutes with $(I-S)$ (and hence also with $(I-S)^{-1}$), so that

$$U^* = (I-S)(I+S)^{-1} = (I+S)^{-1}(I-S) = U^{-1} \Rightarrow U \text{ unitary.}$$



Note

Just as the linear fractional xform $w = \frac{1+z}{1-z}$ maps the imaginary axis onto the unit circle (and the left half plane to its interior), the Cayley transform maps the spectrum of S, which lies on the imaginary axis onto the spectrum of U which lies on the unit circle.

$\langle 2.6 \rangle$ $A = I + uv^*$; Let $A^{-1} = I + \alpha uv^*$. Then

$$AA^{-1} = I + (1+\alpha) uv^* + \alpha (uv^* uv^*)$$

$$= I + (1 + \alpha + (v^*u)) uv^* \quad \text{since } v^*u \text{ is a scalar.}$$

Then $\alpha = \frac{-1}{1+(v^*u)}$ gives the inverse, unless

$v^*u = -1$, when A is singular.

Since uv^* is of rank 1, null(A) must be 1-dimensional ($Au = u + u(v^*u) = u - u = 0$, i.e. $N(A) = \text{span}\{u\}$).

3.3 $A \in \mathbb{C}^{m\times n}$, $x \in \mathbb{C}^m$

(a) $\boxed{\|x\|_\infty \leq \|x\|_2}$

$$\|x\|_\infty = \max_{1\le i\le m} |x_i| = \sqrt{\max_{1\le i\le m} |x_i|^2} \leq \sqrt{\sum_{i=1}^m |x_i|^2} = \|x\|_2$$

(equality achieved for, say, $x = \underline{e}_k$, $k = 1,..,m$

(b) $\boxed{\|x\|_2 \leq \sqrt{m}\,\|x\|_\infty}$

$$\|x\|_2 = \sqrt{|x_1|^2 + .. + |x_m|^2} \leq \sqrt{m * \max_{1\le i\le m} |x_i|^2} = \sqrt{m}\,\max_i |x_i| = \sqrt{m}\,\|x\|_\infty$$

(c) $\|A\|_\infty \leq \sqrt{n}\,\|A\|_2$

$$\|A\|_\infty^2 = \max_{\|x\|_\infty = 1} \|Ax\|_\infty^2 = \max_{i=1,..,m} |a^i x|^2 = |a^I x|^2$$

for $x_j = \text{sgn}(a_{Ij}) = \pm 1$

$$\left(= |a^I|_1^2\right) \leq \sum_{i=1}^m |a^i x|^2 = \|Ax\|_2^2 \leq \|A\|_2^2 \|x\|_2^2$$

$$\Rightarrow \boxed{\|A\|_\infty \leq \sqrt{n}\,\|A\|_2}$$

Note: here x is a vector with values (± 1), chosen so that $a^I x = \sum_{j=1}^n |a_{Ij}|$, the 1-norm of the I-th row (the one with the max. 1-norm).

Now, get $a^1 = (b, b, \ldots, b)$, ~~so that~~ while all other rows vanish. (then $x = (1, .., 1)^T$).

Then $\|A\|_\infty = n|b|$, while

$$AA^H = \begin{pmatrix} b & \overset{n}{—} & b \\ & 0 & \end{pmatrix}_m \begin{pmatrix} b & \\ | & 0 \\ b & \end{pmatrix} = \begin{pmatrix} nb^2 & \\ & 0 \end{pmatrix}$$

so that $\lambda_{max}(AA^H) = \sqrt{n}|b|$ ▲

(d) ~~Ke~~ $\boxed{\|A\|_2 \le \sqrt{m}\,\|A\|_\infty}$

Let x: $\|x\|_2^2 = 1$, with $\|Ax\|_2^2 = \|A\|_2$

(this x is simply the eigenvector of $A^H A$ ~~(or/ AA^H)~~ associated to the largest eigenvalue).

Then:

i-th row

$$\|A\|_2^2 = \|Ax\|_2^2 = \sum_{i=1}^{m} |a^i x|^2$$

But $$|a^i x|^2 = \Big|\sum_{j=1}^{n} a_{ij} x_j\Big|^2 \le \sum_{j=1}^{n} |a_{ij} x_j|^2$$

(since $|x_j| \le 1$, from $\|x\|_2 = 1$) $$\le \Big(\sum_{j=1}^{n} |a_{ij}|\Big)^2 = |a^i|_1^2$$

max row norm

Then $$\sum_{i=1}^{m} |a^i x|^2 \le \sum_{i=1}^{m} |a^i|_1^2 \le m\,|a^I|_1^2 = m\,\|A\|_\infty^2$$

$$\Rightarrow \|A\|_2 \le \sqrt{m}\,\|A\|_\infty$$

Example: Let $A = \begin{pmatrix} b & \\ \vdots & O \\ b & \end{pmatrix}$; $A^H A = \begin{pmatrix} mb^2 & \\ & O \end{pmatrix}$;

$A^H A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = mb^2 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ i.e. largest eval. is mb^2, evec $x = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

$$\left.\begin{aligned} &\text{Then } \|A\|_2 = \sqrt{m}\,b \\ &\text{Also } \|A\|_\infty = |b| \end{aligned}\right\}$$

(3.2) $\rho(A) = |\lambda|_i \Rightarrow \exists\, x : Ax = \lambda x$ i.e. $\frac{1}{5/6}$

$$\left.\begin{array}{l} \|Ax\| = |\lambda|\,\|x\| \\ \text{But } \|Ax\| \leq \|A\|\,\|x\| \end{array}\right\} \Rightarrow |\lambda| \leq \|A\| \quad \text{in any norm}$$

Since $\rho(A) = \max|\lambda| \Rightarrow \rho(A) \leq \|A\|$.

(3.3)(a) $\|x\|_\infty^2 = \max_i |x_i|^2 \leq (|x_1|^2 + \dots + |x_m|^2) = \|x\|_2^2$

(b) $\|x\|_2^2 = (|x_1|^2 + \dots + |x_m|^2) \leq m \max_i |x_i|^2 = m\|x\|_\infty^2$

(c) $\|A\|_\infty^2 = \max_{i=1,..m} \left(\sum_{j=1}^{n} |a_{ij}| \right)^2 \quad \#/\cancel{\|A\|}$

Now let x be the vector with entries ± 1 for which the equality $(\|x\|_\infty = 1)$

$$\|Ax\|_\infty = \|A\|_\infty \|x\|_\infty = \|A\|_\infty \text{ is achieved.}$$

Then, since for any vector x, we have by (a)

$$\begin{array}{l} \|Ax\|_\infty \leq \|Ax\|_2 \leq \|A\|_2 \|x\|_2 = \|A\|_2 \sqrt{n} \\ \quad \| \\ \|A\|_\infty \end{array} \qquad (x \in \mathbb{C}^n)$$

(d) $\|A\|_2^2 = \max_{\|x\|=1} \|Ax\|_2^2 \leq \max_{\|x\|=1} \left\{ \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right)^2 \right\} \leq$

$\leq \max_{\|x\|=1} \sum_{i=1}^{m} \left(\sum_{j=1}^{n} |a_{ij}||x_j| \right)^2 \leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} |a_{ij}| \right)^2 \leq$

$\leq m \cdot \left(\max_{1 \leq i \leq m} \left(\sum_{j=1}^{n} |a_{ij}| \right) \right)^2 = m \cdot \|A\|_\infty^2$

+ 0/6

(3.5) $E = uv^* \Rightarrow \|E\|_2 = \|u\|_2 \|v\|_2$

(Since $\|uv^*\|_2^2 = \max\limits_{\|x\|=1} \|uv^*x\|_2^2 = \max \|u\|_2^2 \, |v^*x|_2^2 \underset{\max}{\leq} \|u\|_2^2 \, \|v^*\|_2^2 \, \|x\|_2^2$

$= \|u\|_2^2 \, \|v^*\|_2^2$

But if $x = \dfrac{v}{\|v\|}$, $\left|v^* \dfrac{v}{\|v\|}\right|_2^2 = \|v\|_2^2$ and r.h.s. is achieved, so claim follows)

Now, consider $\|E\|_F^2 = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} |u_i v_j^*|^2$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (u_i u_i^*)(v_j v_j^*)$$

$$= \sum_{i=1}^{m} |u_i|^2 \sum_{j=1}^{n} |v_j|^2 = \|u\|_2^2 \, \|v\|_2^2 \equiv \|u_F\|^2 \, \|v\|_F^2$$

Since F-norm & 2-norm coincide on vectors.

2.1 "A unitary, triangular $\Rightarrow$ A diagonal"

◁ By problem (1.3), the inverse of an upper-triang. matrix A is also upper triangular.

Since A unitary $\Rightarrow$ $A^{-1} = A^H$ which is lower triangular.

i.e. A^{-1} is both upper and lower triangular, hence it is diagonal, and so is A. ▷

3.2 $\rho(A) \leq \|A\|$, with $\|\cdot\|$ any ~~consistent~~ induced norm

(note: in class we proved that this is true for any subnorm, because one can always associate a vector norm — this problem is slightly easier, because $\|\cdot\|$ is assumed to be defined from a vector norm).

◁ Let (λ, x) be evalue/evector of A: $(x \neq 0)$

$$Ax = \lambda x \Rightarrow \|Ax\| = |\lambda| \|x\| \Rightarrow$$

$$|\lambda| \|x\| \leq \|A\| \|x\| \Rightarrow |\lambda| \leq \|A\|$$

Since this is true for any eigenvalue $\lambda \in \mathcal{L}(A)$,

we have $\max |\lambda| \equiv \rho(A) \leq \|A\|$ ▷

1.3 "R nonsingular, upper triangle $\Rightarrow$ K ~~same~~ also upper triangle."

$$\text{Fix } i \in \{1, \dots, n\}$$

$R = \begin{pmatrix} & \\ 0 & \end{pmatrix}$; clearly $\{r_1, \dots, r_i\}$ are independent and they all belong to the subspace spanned by $\{e_1, \dots, e_i\}$, i.e. $r_j \in \operatorname{span}\{e_k\}_{k=1}^{i}$ with $j = 1, \dots, i$. This subspace is i-dimensional. Since the r_j ~~form~~ are i independent vectors, they form a basis, so that each of the vectors $\{e_k\}_{k=1}^{i}$ can be expressed as a linear combination of the vectors $\{r_k\}_1^i$. More specifically,

$$e_i = \sum_{j=1}^{i} z_{ij} r_j$$

and this is true for all $i = 1, \dots, n$.

But then, it is clear that the array formed by the $\{z_{ij}\}_{j=1}^{i}$, $i = 1, \dots, n$

i.e. $Z = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ & z_{22} & & \vdots \\ & 0 & \ddots & \\ & & & z_{nn} \end{pmatrix}$ is triangular with the property

$$(r_1 \cdots r_n) \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ & z_{22} & & \vdots \\ & & \ddots & \\ & & & z_{nn} \end{pmatrix} = (e_1 \cdots e_n) = I$$

i.e. $Z = R^{-1}$ is upper triangular.