Math 504
S'02, #9

24. (1, 2, 4)

24.1 $\forall$ of the following statements, prove that it is true or give an example to show that it is false. Throughout, $A \in \mathbb{C}^{m\times m}$ unless otherwise indicated.

(a) $\lambda \in sp(A)$, $\mu \in \mathbb{C} \Rightarrow \lambda - \mu \in sp(A - \mu I)$: True

$\Downarrow$

$\det(A - \lambda I) = 0 \Rightarrow \det\left((A-\mu I) - (\lambda - \mu) I\right) = 0$

(b) $A \in \mathbb{R}^{m\times m}$, $\lambda \in sp(A) \Rightarrow -\lambda \in sp(A)$ FALSE

eg: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(c) $A \in \mathbb{R}^{m\times m}$, $\lambda \in sp(A) \Rightarrow \bar{\lambda} \in sp(A)$ TRUE

(Complex roots for real polynomials occur in conj. pairs).

(d) $\lambda \in sp(A)$, A^{-1} exists $\Rightarrow \lambda^{-1} \in sp(A^{-1})$ TRUE

$\det(A - \lambda I) = \det[A(I - \lambda A^{-1})] = \det A \cdot \lambda^{n} \cdot \det(\lambda^{-1} I - A) = 0$

$\Rightarrow \det(A - \lambda^{-1} I) = 0$ since $\det A \neq 0$, $\lambda \neq 0$ (A nonsing.)

(e) If $sp(A) = \{0\} \Rightarrow A = 0$ FALSE

e.g. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(f) $A = A^H$, $\lambda \in sp(A) \Rightarrow |\lambda|$ singular value of A TRUE

($sp(A) \subset \mathbb{R}$; $\lambda \in sp(A) \Rightarrow \lambda^2 \in sp(A^H A) = \sigma^2$, $\sigma > 0$).

(g) A diagonalizable $sp(A) = \{\lambda\} \Rightarrow A$ diagonal TRUE

$(X^{-1} D X = \lambda X^{-1} X = \lambda I = A)$.

24.2 (Gershgorin thm) Let $A \in \mathbb{C}^{n\times n}$, and for $i = 1 \ldots n$ let

$$r_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|.$$ The i-th Gershgorin disk $D_i(A)$ is the set of $z \in \mathbb{C}$: $|z - a_{ii}| \leq r_i$ (closed disk of radius r_i, centered at a_{ii}).

(a) Each evalue λ of A lies within one of the $D_i(A)$.

◁ Let $\lambda \in sp(A)$, $Av = \lambda v$: evector v. Let i be an index such that $|v_i| = \max_{1\leq j\leq n} |v_j|$.

Consider the i-th component of $Av = \lambda v$:

$$\lambda v_i = a_{ii} v_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} v_j$$

$$\Rightarrow |\lambda - a_{ii}||v_i| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}||v_j| \leq \Big(\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|\Big)|v_i| = r_i |v_i|$$

$$\Rightarrow |\lambda - a_{ii}| \leq r_i \quad (v_i \neq 0 \text{ else } v \equiv 0).$$ ▷

(b) Consider $D = \operatorname{diag}(a_{ii})$; $B := A - D$.

Introduce $A(\varepsilon) = D + \varepsilon B$ $(A(0) = D,\ A(1) = A)$.

Now $sp(A(0)) = \{a_{ii} \mid i = 1, \ldots, n\}$ and for sufficiently small ε all G-disks of $A(\varepsilon)$ are disjoint.

Since the eigenvalues of a matrix whose entries vary continuously with a parameter ε are continuous functions of that parameter, we

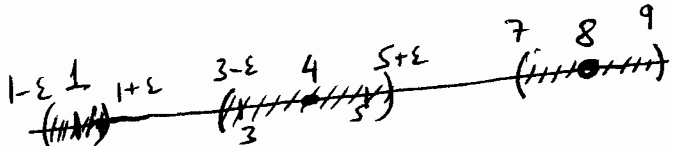
Conclude that the i-th evalue of $A(\varepsilon)$ lies in the disk centered at a_{ii} with radius εr_i.

If, as $\varepsilon \uparrow$, k some disks intersect, then the k evalues associated with the original (disjoint) k disks will lie somewhere in their union; on the other hand, any disks remaining isolated as $\varepsilon \to 1$ will contain a single evalue. ▮

(c) $A = \begin{pmatrix} 8 & 1 & 0 \\ 1 & 4 & \varepsilon \\ 0 & \varepsilon & 1 \end{pmatrix}$, $|\varepsilon| < 1$; ($A = A^H$ so spectrum real)

then $D_1 = \{ z \mid |8-z| \leq 1 \}$, $D_2 = \{ z : |4-z| \leq 1+\varepsilon \}$

$D_3 = \{ z : |z-1| \leq \varepsilon \}$.



(d) Let $D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$; $A \sim D^{-1} A D = \begin{pmatrix} 8 & b/a & 0 \\ a/b & 4 & \varepsilon(c/b) \\ 0 & \varepsilon(b/c) & 1 \end{pmatrix}$

Now $D_3' = \{ z : |z-1| \leq \varepsilon (b/c) \}$

Choosing $b/c = \varepsilon$ ($|\varepsilon| < 1$), $\Rightarrow D_3' = \{ z : |z-1| \leq \varepsilon^2 \}$;

Since $D_2' = \{ z : |4-z| \leq |a/b + 1| \}$, if we let $a = b = \varepsilon$, $c = 1$; then the disks remain disjoint.

For $A \in \mathbb{C}^{m\times m}$, $\|\cdot\|$ any norm:

(a) $\lim_{n\to\infty}\|A^n\|=0 \Leftrightarrow \rho(A)<1$ (b) $\lim_{t\to\infty}\|e^{tA}\|=0 \Leftrightarrow \alpha(A)<0$

$\overset{(b)}{\Rightarrow}\left(\rho(e^{tA}) = \max|e^{t\lambda_i}| = e^{t\,\mathrm{Re}(\lambda)_{max}}\right)$

<u>24.4</u> $A \in \mathbb{C}^{m\times m}$, $\|\cdot\|$ arbitrary;

(a) $A = Q^*TQ \Rightarrow A^n = Q^*T^nQ$

Converse easy, since $\rho(A)\le\|A\|$, so $\|A\|<1 \Rightarrow \rho(A)<1$ ↳ convergence.

($\Leftarrow$) Let $\rho(A)<1 \Rightarrow |\lambda_{max}| = \rho(A) < 1 \Rightarrow$ $\rho(A^n) = |\lambda_{max}|^n \longrightarrow 0$. To show that this implies $A^n \to 0$ we must show that in some norm: $\|A\|<1$. Now let Q be such that

$$QAQ^* = T = \begin{pmatrix} \lambda_1 & & \ast \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \quad \text{and} \quad D = \mathrm{diag}(s, s^2, \dots, s^n)$$

so that
$$DTD^{-1} = \begin{pmatrix} \lambda_1 & s^{-1}t_{12} & s^{-2}t_{13} & \cdots & s^{-n+1}t_{1n} \\ & \lambda_2 & s^{-1}t_{23} & \cdots & s^{-n+2}t_{2n} \\ & & \lambda_3 & & \vdots \\ & 0 & & \ddots & s^{-1}t_{n-1,n} \\ & & & & \lambda_n \end{pmatrix}$$

For s sufficiently large the maximum absolute row sum will be found in the row containing the largest λ_i, and it will be $\|DTD^{-1}\|_\infty \le \lambda_{max} + s^{-1}n|t_{ij\,max}| < \lambda_{max}+\varepsilon$ for any ε. Then, if $\lambda_{max} = \rho(A) \le 1-\varepsilon < 1$, the norm

$$\|B\|_s := \|DQBQ^*D^{-1}\|_\infty$$

Gives $\|A\|_s = \|DTD^{-1}\|_\infty \le \lambda+\varepsilon < 1$ and $\|A^n\| \le \|A\|^n \to 0$

(4) Let A have the block tridiagonal form

$$\mathcal{A} \equiv [B_i, A_i, C_i]$$

where each A_i, B_i and C_i has order m and $i = 1, 2, \ldots, n$ so that $\mathcal{A}$ has order mn. Show that reduction to the block-factored form $\mathcal{L}\mathcal{U} = \mathcal{A}$ (with $\mathcal{L}$ block lower triangular and $\mathcal{U}$ block upper triangular) without pivoting is possible if in any induced norm:

(i) A_i are nonsingular for $1 \leq i \leq n$;

(ii) $\|A_i^{-1}B_i\| + \|A_i^{-1}C_i\| \begin{cases} \leq 1, & 1 \leq i \leq n \\ < 1, & \text{for some } i \end{cases}$

and

(iii) $C_i \not\equiv O_{n,n}$ for $1 \leq i \leq n-1$

and $B_i \not\equiv O_{nn}$ for $2 \leq i \leq n$.

A We assume that it is possible to have a decomposition of the form

$$\begin{pmatrix} A_1 & C_1 & & O \\ B_2 & A_2 & C_2 & \\ & \ddots & \ddots & C_{n-1} \\ O & & B_n & A_n \end{pmatrix} = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \ddots & & O \\ & \ddots & \ddots & \\ O & & \beta_n & \alpha_n \end{pmatrix} \begin{pmatrix} I & \gamma_1 & & O \\ & \ddots & \ddots & \\ & & \ddots & \gamma_{n-1} \\ O & & & I \end{pmatrix}$$

where $\alpha_1 = A_1$, $\gamma_1 = A_1^{-1} C_1$

$$\alpha_i = A_i - B_i \gamma_{i-1} \, , \quad \gamma_i = \alpha_i^{-1} C_i$$

$$i = 2, \dots, n \qquad\qquad i = 2, \dots, n-1$$

Given that (i) A_i nonsingular & (iii) $\|A_i^{-1} B_i\| + \|A_i^{-1} C_i\| \leq 1$

(ii) $B_i, C_i \equiv O_{nn}$ in any norm (with $<$ true once at least)

Must show that α_i^{-1} exists: (by induction)

<u>$i=1$</u> : $\begin{cases} \alpha_1 = A_1 , & \text{nonsingular by assumption} \\ \gamma_1 = A_1^{-1} C_1 & \Rightarrow \|\gamma_1\| = \|A_1^{-1} C_1\| = \\ & = \|A_1^{-1} C_1\| + \|A_1^{-1} \overset{(\equiv 0)}{B_1}\| \leq 1 \end{cases}$

~~$i \neq 2$~~

<u>$i=2$</u> . $\alpha_2 = A_2 - B_2 \gamma_1 = A_2 \left(I - A_2^{-1} B_2 \gamma_1 \right)$

Now $I + B$ nonsingular if $\|B\| < 1$ (Bannach lemma)

Here $\|A_2^{-1} B_2 \gamma_1\| \leq \|A_2^{-1} B_2\| \|\gamma_1\| \leq \|A_2^{-1} B_2\|$

$< \|A_2^{-1} B_2\| + \|A_2^{-1} C_2\|^{\circledast} \leq 1$

$\Rightarrow \|A_2^{-1} B_2 \gamma_1\| < 1$

$\Rightarrow \alpha_2$ nonsingular

$\circledast$ $C_2 \neq 0 \Rightarrow \|A_2^{-1} C_2\| > 0$)

For $2<k<n$, proceed similarly:

assume $\|\gamma_{k-1}\| \leq 1$, α_{k-1} nonsingular. We prove that $\{\alpha_k \text{ nonsingular}, \|\gamma_k\|<1\}$

◀ Indeed, $\alpha_k = A_k - B_k\gamma_{k-1} = A_k(I - A_{k-1}^{-1}B_k\gamma_{k-1})$

$\Rightarrow \alpha_k$ nonsingular if $\|A_k^{-1}B_k\gamma_{k-1}\|<1$ (in any induced norm) (again Bannach lemma)

But $\|A_k^{-1}B_k\gamma_{k-1}\| \leq \|A_k^{-1}B_k\|\|\gamma_{k-1}\| \leq \|A_k^{-1}B_k\|$

$< \|A_k^{-1}B_k\| + \|A_k^{-1}C_k\| \leq 1$

$\hookrightarrow (C_k \neq 0)$

$\Longrightarrow \|A_k^{-1}B_k\gamma_{k-1}\| < 1$, as we wanted,

and $\gamma_k = \alpha_k^{-1}C_k = (I - A_k^{-1}B_k\gamma_{k-1})^{-1}A_k^{-1}C_k$

Then $\|\gamma_k\| \leq \|(I - A_k^{-1}B_k\gamma_{k-1})^{-1}\|\|A_k^{-1}C_k\| \leq$

$$\leq \frac{\|A_k^{-1}C_k\|}{1-\|A_k^{-1}B_k\gamma_{k-1}\|} \leq \frac{\|A_k^{-1}C_k\|}{1-\|A_k^{-1}B_k\|\|\gamma_{k-1}\|}$$

$$\leq \frac{\|A_k^{-1}C_k\|}{1-\|A_k^{-1}B_k\|} \leq 1$$ (Because of condition (ii))

(Here $1-\|A_k^{-1}B_k\gamma_{k-1}\| \geqslant 1-\|A_k^{-1}B_k\|\|\gamma_{k-1}\|$
$\geqslant 1-\|A_k^{-1}B_k\| \geqslant \|A_k^{-1}C_k\|$).

Note that if either $\|\gamma_{k-1}\| < 1$ or if $\|A_k^{-1}B_k\| + \|A_k^{-1}C_k\| < 1$ we'd have gotten $\|\gamma_k\| < 1$ so if this inequality is strict for some k, then $\|\gamma_j\| < 1$ strictly for all $j \geq k$.

Then, by our hypothesis, "either $\boxed{\|\gamma_{n-1}\| < 1}$ or, ~~since~~ if $\|\gamma_{n-1}\| = 1$, this would mean that the inequality ($\|\gamma_k\| \leq 1$) was actually an equality for all $i < n$. By our hypothesis, then, it would follow that

$$\|A_n^{-1}B_n\| + \|A_n^{-1}\underbrace{C_n}_{\equiv 0}\| < 1$$

(one of them at least must be < 1)

$$\Rightarrow \boxed{\|A_n^{-1}B_n\| < 1.}$$

In either case,

$$\alpha_n = A_n - B_n\gamma_{n-1} = A_n\left(I - A_n^{-1}B_n\gamma_{n-1}\right)$$

is nonsingular since

$$\|A_n^{-1}B_n\gamma_{n-1}\| \leq \|A_n^{-1}B_n\| \, \|\gamma_{n-1}\| < 1$$

and the proof is complete

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S'02, #10

$f(A)$; 25.1, 27.2

(1) The field of values of $A \in \mathbb{C}^{m \times m}$ is defined as $\mathcal{F}(A) := \{z \mid z = x^H A x,\ \|x\| = 1\}$

Establish

(a) $\mathcal{F}(\alpha A + \beta I) = \alpha \mathcal{F}(A) + \beta$

(i) $z \in \mathcal{F}(\alpha A + \beta I) \Rightarrow \underset{(\|x\|=1)}{\exists x}: \ x^H(\alpha A + \beta I)x = z = \alpha(x^H A x) + \beta$

$\Rightarrow z \in \alpha \mathcal{F}(A) + \beta \quad \Rightarrow \mathcal{F}(\alpha A + \beta I) \subseteq \alpha \mathcal{F}(A) + \beta$

(ii) $z \in \alpha \mathcal{F}(A) + \beta \Rightarrow \exists x : \ \alpha(x^H A x) + \beta = z$; but then

$x^H(\alpha A + \beta I)x = \alpha(x^H A x) + \beta x^H x = z \Rightarrow z \in \mathcal{F}(\alpha A + \beta I)$

(b) $\overline{\underline{Sp(A) \subseteq \mathcal{F}(A)}}$: $\lambda \in sp(A) \Rightarrow \exists v : \ Av = \lambda v$

$\Rightarrow \underline{v^H A v = \lambda v^H v = \lambda} \Rightarrow \lambda \in \mathcal{F}(A)$

(c) U unitary $\Rightarrow \overline{\underline{\mathcal{F}(U^H A U) = \mathcal{F}(A)}}$:

$z \in \mathcal{F}(U^H A U) \Rightarrow \exists x : \ z = x U^H A U x = (Ux)^H A (Ux)$

$= y^H A y \Rightarrow z \in \mathcal{F}(A)$

(and the converse follows trivially)

(d) $\mathcal{F}(A+B) \subseteq \mathcal{F}(A) + \mathcal{F}(B)$

let $z \in \mathcal{F}(A+B) \Rightarrow \exists x : \ x^H(A+B)x = z \Rightarrow$

$\Rightarrow x^H A x + x^H B x = z \Rightarrow z \in \mathcal{F}(A) + \mathcal{F}(B)$

25.1 (a) Let $A \in \mathbb{C}^{m\times m}$ be tridiagonal & hermitian, with all the off-diagonal entries $\neq 0$. Prove that the evalues of A are distinct.

(a) By the Separation thm, if A_{n-1} is the $(n-1)$-order principal minor, and $\{\lambda_k'\}_1^{n-1} \in sp(A_{n-1})$, $\{\lambda_k\}_{k=1}^{n} \in sp(A)$, we have

$$\lambda_k \leq \lambda_k' \leq \lambda_{k+1} \qquad (*)$$

As proved in class, the characteristic polynomials of the principal minors, $P_k(\lambda) = \det(A_k - \lambda I_k)$ satisfy the 3-term recurrence

$$P_{k+1}(\lambda) = (a_{k+1} - \lambda) P_k(\lambda) - |b_k|^2 P_{k-1}(\lambda)$$

so that, since $|b_k| \neq 0$, no two polynomials of successive orders can have a common zero.

This implies that the inequalities in $(*)$ are strict $\Rightarrow$ the evalues of A (zeros of $P_n(\lambda)$) are distinct.

(b) Let $A = \begin{pmatrix} 1 & & 0 \\ 1 & \ddots & \\ 0 & 1 & 1 \end{pmatrix}$; A is upper Hessenberg, satisfying requirements (i.e. subdiags. $\neq 0$) but $sp(A) = \{1\}$.

27.2 ~~[illegible]~~, Let $A \in \mathbb{C}^{m\times m}$ be arbitrary.

(a) Show $\mathcal{F}(A)$ contains the convex hull of the eigenvalues of A

As shown in class, $\mathcal{F}(A) \subset \mathbb{C}$ is convex, and we know that $Sp(A) \subset \mathcal{F}(A)$. Since a convex set contains (by definition) the convex hull of any of its subsets, proof follows

(b) If A is normal ($AA^H = A^H A$), then

$$\mathcal{F}(A) = \mathcal{CH}(sp(A)) \quad \left(\text{the convex hull of the eigenvalues of } A\right).$$

~~Let $\sigma, \rho \in \mathcal{F}(A) \Rightarrow \exists\, x, y : x^H A x = \sigma,\ y^H A y = \rho$.~~

~~We must~~ If A is normal the eigenvectors form an orthonormal set. ~~Then~~ Let $x = \sum_{i=1}^{n} \alpha_i v_i$; then

$$x^H A x = \left(\sum \alpha_i^H v_i^H\right)\left(\sum \alpha_i \lambda_i v_i\right) = \sum \lambda_i |\alpha_i|^2$$

The latter expression represents a weighted average of the λ_i (since $\sum |\alpha_i|^2 = 1$) and any such combination is possible. Thus $\mathcal{F}(A)$ coincides with the ~~field of values~~ convex hull of $sp(A)$.