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Canonical Forms and Eigenspace Computations for Matrix Pencils with Applications

PROJECT PLAN

Summary. This project is a prolongation of the TFR project 222–95–34. The long-term goal is to develop improved algorithms for computing the Jordan and Kronecker structures for given matrices and matrix pencils. In several applications (e.g., in control theory) it is also important to know which nearby Jordan and Kronecker structures exist. Our algorithms and software tools will be delivering such information based on the combinatorial relationships among the various Jordan and Kronecker structures known as *strata.* Building on our earlier and more recent contributions regarding algorithms, software, applications and mathematical theory, we now aim at solving these problems for both dense and sparse matrices and matrix pencils, including small to medium-sized problems as well as large-scale problems. Our industrial partners in the thematic network NICONET [42] funded by the European Commission contribute with new interesting applications (e.g., in automotive, aerospace, petro-chemical and glass industry). The acronym stands for *Network for Development and Evaluation of Numerically Reliable Software in Control Engineering and its Implementation in Production Technologies.*

The project plan is organized as follows. We start by summarizing some mathematical background, including Jordan and Kronecker structures, generalized Schur form, generic and degenerate pencils, and mention some applications. Then we summarize the state-of-the-art regarding staircase-type algorithms for computing the generalized Schur form (or GUPTRI form), and formulate different "nearness" problems to be considered. Next, the geometry of matrix and matrix pencil spaces and some of our recent contributions regarding the problem of *stratification* of orbits and bundles of matrices and matrix pencils [21, 16], and *versal deformations* for matrix pencils [15] are reviewed. Our objective is to make use of this knowledge to improve the understanding about numerical algorithms and their failures. Finally, we summarize the most recent results within this project and discuss how to proceed in developing more robust and accurate algorithms and software for computing the (partial) generalized Schur form of a general matrix pencil and increase their functionality in order to be able solve interesting "nearness" problems. Throughout, the paper we also refer to other interesting work that relate to our project.

Mathematical Background and Some Applications. Singular matrix pencils $A - \lambda B$, where A and B are m-by-n matrices with real or complex entries, appear in several applications. Examples include problems in control theory relating to a linear system $E\dot{x}(t) = Fx(t) + Gu(t)$, where E and F are p-by-p matrices, and G is p-by-k. Solvability issues of a descriptor or singular system (i.e., $\det(E) = 0$), such as the existence of a solution, consistent initial values, and its explicit solution can be revealed from the Kronecker structure of $A - \lambda B \equiv F - \lambda E$ (e.g., see [23, 56]). The problems to find the controllable subspace, uncontrollable modes or an upper bound on the distance to uncontrollability for a controllable system $E\dot{x}(t) = Fx(t) + Gu(t)$ can all be formulated and solved in terms of certain reducing subspaces of the matrix pencil $A - \lambda B \equiv [G \quad F] - \lambda[0 \quad E]$ (e.g., see [43, 50, 51, 11]).

In most applications it is sufficient to transfer $A - \lambda B$ to a *generalized Schur form* (e.g., to

GUPTRI form [12, 13])

$$P^H(A - \lambda B)Q = \begin{bmatrix} A_r - \lambda B_r & * & * \\ 0 & A_{reg} - \lambda B_{reg} & * \\ 0 & 0 & A_l - \lambda B_l \end{bmatrix}, \tag{1}$$

where P (m-by-m) and Q (n-by-n) are unitary and $*$ denotes arbitrary conforming submatrices. Here the square upper triangular block $A_{reg} - \lambda B_{reg}$ is regular and has the same regular structure as $A - \lambda B$ (i.e., contains all generalized eigenvalues (finite and infinite) of $A - \lambda B$). The rectangular blocks $A_r - \lambda B_r$ and $A_l - \lambda B_l$ contain the singular structure (right and left minimal indices) of the pencil and are block upper triangular. The *singular blocks of right* (column) and *left* (row) *indices of grade* j are

$$L_j \equiv \begin{bmatrix} -\lambda & 1 & & \\ & \cdot & \cdot & \\ & & -\lambda & 1 \end{bmatrix} \quad \text{and} \quad L_j^T \equiv \begin{bmatrix} -\lambda & & \\ 1 & \cdot & \\ & \cdot & -\lambda \\ & & 1 \end{bmatrix}$$

of size j-by-$(j+1)$ and $(j+1)$-by-j, respectively. $A_r - \lambda B_r$ has only right minimal indices in its Kronecker canonical form (KCF), indeed the same L_j blocks as $A - \lambda B$. Similarly, $A_l - \lambda B_l$ has only left minimal indices in its KCF, the same L_j^T blocks as $A - \lambda B$. If $A - \lambda B$ is singular at least one of $A_r - \lambda B_r$ and $A_l - \lambda B_l$ will be present in (1). The explicit structure of the diagonal blocks in staircase form can be found in [13]. If $A - \lambda B$ is regular $A_r - \lambda B_r$ and $A_l - \lambda B_l$ are not present in (1) and the GUPTRI form reduces to the upper triangular block $A_{reg} - \lambda B_{reg}$. Staircase forms that reveal the Jordan structure of the zero and infinite eigenvalues are contained in $A_{reg} - \lambda B_{reg}$.

Given $A - \lambda B$ in GUPTRI form we also know different pairs of reducing subspaces [51, 12]. Suppose the eigenvalues on the diagonal of $A_{reg} - \lambda B_{reg}$ are ordered so that the first k, say, are in Λ_1 (a subset of the spectrum of $A_{reg} - \lambda B_{reg}$) and the remainder are outside Λ_1. Let $A_r - \lambda B_r$ be m_r-by-n_r. Then the *left and right reducing subspaces* associated with Λ_1 are spanned by the leading $m_r + k$ columns of P and the leading $n_r + k$ columns of Q, respectively. When Λ_1 is empty, the corresponding reducing subspaces are called *minimal*, and when Λ_1 contains the whole spectrum the reducing subspaces are called *maximal*.

If $A - \lambda B$ is m-by-n, where $m \neq n$, then for almost all A and B it will have the same KCF, depending only on m and n (the *generic case*). The generic Kronecker structure for $A - \lambda B$ with $d = n - m > 0$ is

$$\operatorname{diag}(L_\alpha, \ldots, L_\alpha, L_{\alpha+1}, \ldots, L_{\alpha+1}), \tag{2}$$

where $\alpha = \lfloor m/d \rfloor$, the total number of blocks is d, and the number of $L_{\alpha+1}$ blocks is $m \bmod d$ (which is 0 when d divides m) [49, 9]. The same statement holds for $d = m - n > 0$ if we replace $L_\alpha, L_{\alpha+1}$ in (2) by $L_\alpha^T, L_{\alpha+1}^T$. Square pencils are generically regular, i.e., $\det(A - \lambda B) = 0$ if and only if λ is an eigenvalue. The generic singular pencils of size n-by-n have the Kronecker structures [52]:

$$\operatorname{diag}(L_j, L_{n-j-1}^T), \quad j = 0, \ldots, n-1.$$

In summary, generic rectangular pencils have only trivial reducing subspaces and no generalized eigenvalues at all. Generic square singular pencils have the same minimal and maximal reducing subspaces. Only if $A - \lambda B$ satisfies a special condition (lies in a particular manifold) does it have nontrivial reducing subspaces and generalized eigenvalues (the *nongeneric* or *degenerate case*). Moreover, only if it is perturbed so as to move continuously within that manifold do its reducing subspaces and generalized eigenvalues also move continuously and satisfy interesting error bounds [10, 12]. These requirements are natural in many control and systems theoretic problems such as computing controllable subspaces and uncontrollable modes [50, 11].

Staircase Algorithms for Computing a Generalized Schur Form. Several authors have proposed (staircase–type) algorithms for computing a (generalized) Schur form (e.g., see [25, 37, 3, 39, 40, 33, 34, 49, 56]). They are numerically stable in the sense that they compute the exact Kronecker structure (generalized Schur form or something similar) of a nearby pencil $A' - \lambda B'$. Let $\| \cdot \|_E$ denote the Frobenius matrix norm. Then $\delta \equiv \|(A - A', B - B')\|_E$ is an upper bound on the distance to the closest $(A + \delta A, B + \delta B)$ with the KCF of (A', B'). Robust software with error bounds for computing the GUPTRI form of a singular $A - \lambda B$ has been published in [12, 13].

The existing algorithms do not guarantee that the computed generalized Schur form $S - \lambda T$ is the "most" nongeneric Kronecker structure within distance δ, nor that $S - \lambda T$ is the closest pencil with the computed structure. It may exist a closer matrix pencil in the orbit of $S - \lambda T$. Recently, Edelman and Ma gave a geometrical explanation of staircase algorithms failures [17]. However, if δ is of the size $O(\|(A, B)\|_E \epsilon)$, where ϵ is the relative machine precision, we know that (A, B) is close to a matrix with the Kronecker structure that the algorithm reports.

It would of course be desirable to have algorithms that could solve the following "nearness" problems:

- Compute the closest nongeneric pencil of a generic $A - \lambda B$.
- Compute the closest matrix pencil with a specified Kronecker structure.
- Compute the most nongeneric pencil within a given distance δ.
- Compute the closest neighbors (covering pencils) of a given $A - \lambda B$.

If the closest structure is not unique we are mainly interested in the most nongeneric KCF. From the perturbation theory for singular pencils [10] we know that all these problems are ill-posed in the sense that the generalized eigenvalues and reducing subspaces for a nongeneric $A - \lambda B$ can change discontinuously as a function of A and B. Therefore, to be able to solve these problems we need to regularize them by restricting the allowable perturbations as mentioned above.

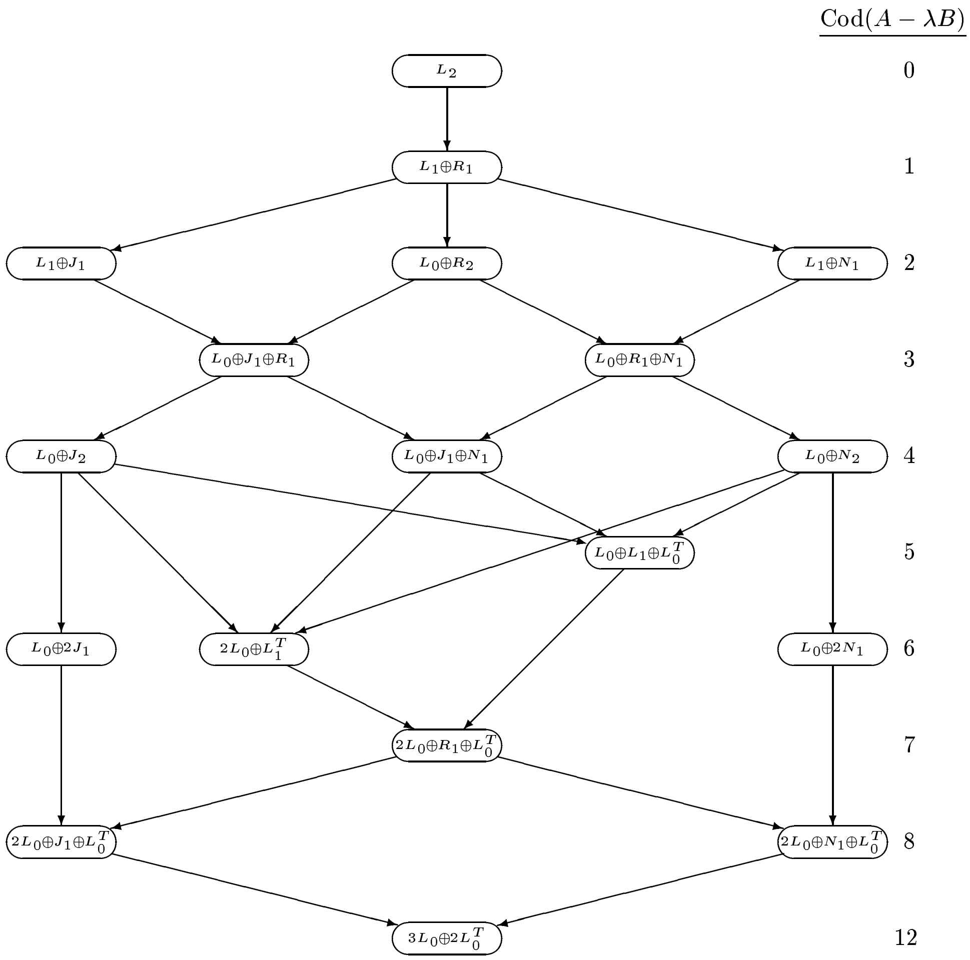
The Geometry of Matrix and Matrix Pencil Spaces. By viewing an n-by-n matrix A as a point in n^2-space instead of an array of numbers or an operator on vectors, and similarly viewing an $m \times n$ matrix pencil $A - \lambda B$ as a point in $2mn$-space, certain numerical computations relating to the Jordan or Kronecker canonical forms can be viewed as moving matrices or matrix pencils from point to point or manifold to manifold in the n^2- and $2mn$-spaces, respectively. Our *objective* is to make use of the geometry of the matrix and matrix pencil spaces, to improve our knowledge of numerical algorithms and their failures. Moreover, with this new knowledge our *goal* is to develop more robust and accurate algorithms (and software) for computing the generalized Schur form of a general matrix pencil and increase their functionality in order to be able to solve some of the "nearness" problems mentioned above.

Following Arnold [2], a *stratified manifold* is the union of nonintersecting manifolds whose closure is the finite union of itself with strata of smaller dimensions (thereby defining stratified manifolds recursively). For matrices, the strata are orbits of similar matrices, or perhaps the union of all orbits with the same Jordan structure and number of distinct eigenvalues including multiplicities (known as *bundles*). For pencils, the strata consist of strictly equivalent pencils (or bundles). The *problem of stratification* is to find the closure relations among the various orbits or bundles, i.e., the closure hierarchy of Jordan and Kronecker structures. These relations define a partial ordering on orbits or bundles. One structure covers another if its closure includes the closure of the other and there is no other structure in between. We use the notation $\mathcal{O}(\cdot)$ and $\mathcal{B}(\cdot)$ for an orbit and bundle, respectively, and $\overline{\mathcal{O}}(\cdot)$ and $\overline{\mathcal{B}}(\cdot)$ denote their closures.

The dimension of an orbit is equal to the dimension of its tangent space. The normal space of an orbit at a certain point (matrix or matrix pencil) is the space complementary and orthogonal to the tangent space at this point. The dimension of the normal space is the codimension of the orbit ($\mathrm{cod}(\cdot)$) which depends only on the Jordan and Kronecker structures (e.g., see [9]). For the orbit of a generic $n \times n$ matrix or matrix pencil we have $\mathrm{cod}(\cdot) = n$ (n specified and distinct eigenvalues). The orbit of a generic rectangular matrix pencil spans the complete space, i.e., $\mathrm{cod}(\cdot) = 0$. All degenerate rectangular cases have $\mathrm{cod}(\cdot) > 0$.

In the next three sections we summarize some of our previous results within this project that form a foundation for the work proposed.

FIG. 1. *A graph displaying the closure hierarchy of the orbits of all* 18 *different KCF for the set of* 2*-by-*3 *matrix pencils.*



The Set of 2-by-3 Matrix Pencils. A comprehensive study of the set of 2-by-3 pencils, the smallest nontrivial rectangular case is published in [21]. The objective was to get a greater understanding of (i) some of the before-mentioned "nearness" problems and how to solve them, and (ii) existing algorithms/software for computing the Kronecker structure of a singular pencil.

The set of 2-by-3 matrix pencils $A - \lambda B$ comprises 18 structurally different Kronecker structures (canonical forms). We show how all the nongeneric structures can be generated by a staircase-type algorithm, starting from the generic canonical form (one L_2 block). Moreover, the algebraic and geometric characteristics of the generic and the 17 nongeneric cases are examined in full detail.

All 2-by-3 pencils "live" in a 12-dimensional space spanned by $\mathcal{O}(L_2)$ and all other structures are in $\overline{\mathcal{O}}(L_2)$. It is just as obvious that $3L_0 \oplus 2L_0^T$ (the zero pencil) is in the closure of the orbit of any other KCF. All other closure relations are not equally obvious (see the closure hierarchy in Figure 1). Necessary conditions on perturbations for transiting from the orbit of one Kronecker structure to another in the closure hierarchy are also derived.

Computable normwise bounds for the smallest perturbations $(\delta A, \delta B)$ of a generic 2-by-3 pencil $A - \lambda B$ such that $(A + \delta A) - \lambda(B + \delta B)$ has a specific nongeneric Kronecker structure are presented. computed Moreover, a study of the behaviour of the nongeneric structures under random perturbations in finite precision arithmetic (using the GUPTRI software [12, 13]) show for which sizes of perturbations the structures are invariant and also that structure transitions occur in accordance with the closure hierarchy. Finally, some of the results are extended to the general m-by-$(m+1)$ case.

Stratification of Orbits and Bundles. Within this project we have recently published new results on the problem of stratification of bundles and orbits of matrices and matrix pencils [16, 19]. We discuss and complete the mathematical theory and propose that knowledge of the closure relations may be applied in the staircase algorithm.

While we are the first to propose the use of stratifications in an algorithm, some of the mathematical theory, at least for nilpotent matrices, goes back to 1961. Combinatorically it is trivial; it is the well-known dominance ordering on partitions (which can be interpreted in terms of the structure indices computed by a staircase algorithm). This is the case of relevance in an algorithm when the eigenvalues are well-clustered so that we may shift all the blocks to be nilpotent.

When eigenvalues are not well clustered, then we have to consider the bundle case as defined by Arnold [2]. We derive new results including the covering and closure relations for this case. We show that testing the closure relation for bundles leads to an NP-complete problem, therefore it may be expensive to use a stratification-enhanced algorithm in the bundle case when more than only a few eigenvalues need to be clustered.

For orbits of matrix pencils, the closure ordering was published in a linear algebra journal by Pokrzywa in 1986 [44], later reformulated by De Hoyos [8]. Recently, Gracia and De Hoyos present safety neighborhoods for the necessary conditions in the change of the Jordan and Kronecker canonical structures under small perturbations [27, 26]. A general unifying algebraic theory of degenerations has been obtained for quivers by a number of authors including Abeasis and del Fra [1] and Bongartz [5]. Bongartz studied the pencil case in 1990, apparently unaware of Pokrzywa's work. In algebraic language, for a matrix or pencil orbit or bundle to cover another, it is necessary to have an extension. This condition is not sufficient; we extend Pokrzywa's results by providing both necessary and sufficient conditions for one Kronecker orbit to cover another. The covering and closure relations for the Kronecker bundle case are also new results.

We summarize the main theorems from [16] in the box below. The closure decision problem is the question of how to test whether the closure of a given orbit or bundle is contained within the closure of another. The closely related covering relationship tells us which structures are covered by a given structure. The "*" indicates that to the best of our knowledge the results were either previously unknown (Theorems 2.3, 2.4, 3.3) or not strong enough for purposes of numerical computations (Theorem 3.2).

	Closure Decision Problem	Covering Relationship
Jordan Orbits	Theorem 2.2	Corollary 2.1
Jordan Bundles	Theorem 2.3, 2.4 *	Theorem 2.3 *
Kronecker Orbits	Theorem 3.1	Theorem 3.2 *
Kronecker Bundles	Theorem 3.3 *	Theorem 3.3 *

The stratification of orbits of pencils associated with problems in control theory has recently been studied by several authors, including Hinrichsen and O'Halloran [30], Boley [4], and Garcia-Planas [24]. For control applications, the pencils of interest typically have no row indices or no column indices. Our results listed above concern general matrix pencils.

Versal Deformations of Matrix Pencils. Is it possible to move smoothly between orbits in the closure hierarchy? The answer is yes and the instrument is the concept of versal deformations. In 1971, V. I. Arnold derived a versal deformation of the Jordan normal form for matrices [2]. Within this project we have generalized and extended the concept of versal deformations to matrix pencils (matrix pairs) and derived a normal form to which not only one specific matrix pencil, but an arbitrary family of matrix pencils close to it can be reduced to by means of a mapping smoothly depending on the elements of the matrix pairs [18, 15]. Another result from this study is a new singular value bound for the distance to the orbits of less generic pencils.

The versal deformation for matrix pencils is more complicated than for matrices since the Kronecker canonical form may consist of Jordan blocks for both finite and infinite eigenvalues as well as Kronecker blocks for right and left minimal indices. For $A - \lambda B$ in Kronecker canonical form we show how to compute the tangent space of the orbit and a parameterized deformation in the normal space. The versal deformation is obtained by adding this parametrized deformation to $A - \lambda B$ and it spans the complete $2mn$-space. The normal form is a miniversal deformation with one parameter for each dimension in the normal space.

The versal deformations reveal the local perturbation theory of matrix pencils related to the KCF. We demonstrate with numerical experiments how this theory may be used in practice to see how computations are influenced by the geometry of the matrix pencil space. For example, we illustrate how small perturbations in tangent and normal spaces, respectively, affect the KCF computed by a staircase algorithm. We also illustrate the connection between versal deformations and the Kronecker structure hierarchy (stratification of matrix pencil orbits) by examples that expose how small perturbations in different directions in the normal space result in different more generic Kronecker structures. We also show that the perturbations required in other directions for finding less generic pencils normally must be significantly larger.

Our Most Recent Results and Proposed Work: Towards Improved Algorithms via Better Understanding of Existing Algorithms and Their Failures. In this section we summarize our recent results within this project that have not been described above. The most recent publications include [6, 7, 16, 20, 31, 32, 35, 38, 53, 54, 55]. Implicitly in this presentation we also outline future work in this project.

Matlab interface to GUPTRI. A Matlab interface to GUPTRI in terms of a MEX-file and some auxiliary routines is now made available (a sample usage and a description of the parameters can be found in [35]):

$$[\mathtt{S},\mathtt{T},\mathtt{P},\mathtt{Q},\mathtt{kstr}] = \mathtt{guptri}(\mathtt{A},\mathtt{B},\mathtt{EPSU},\mathtt{GAP},\mathtt{ZERO})$$

StratiGraph – a tool for computing and presenting hierarchies of Jordan and Kronecker structures. StratiGraph is a Java-based tool for computation and presentation of graphs displaying closure hierarchies of Jordan and Kronecker structures [31, 32, 20]. StratiGraph builds on our results concerning the stratification of orbits and bundles of matrices and matrix pencils [21, 16]. Given the dimension of the problem and a canonical structure, the user can easily get information about nearby Jordan and Kronecker structures in the closure hierarchy.

Any existing algorithm for the computation of the Jordan or Kronecker canonical form of a matrix or pencil proceeds in stages. Let us illustrate with the matrix case. The most generic matrices are nonsingular and diagonalizable, so at the start of the computation the best possible guess is that the matrix will be diagonalizable. This notion is subject to modification as more information from the computation is obtained. At each step in the staircase algorithm we learn that the matrix is in the closure of the set of matrices similar to some other less generic Jordan form. Indeed, one may view the algorithm as identifying a nested sequence of closures. If during the course of the algorithm, the user is unhappy with any choice, he/she might wish to have the power to backtrack through the algorithm and be offered other choices. Alternatively at the end of the algorithm the user might wish to know what he/she has missed in the “thicket” of nearby structures.

As a direct application, since we expect (staircase) algorithms to compute Kronecker structures that are nearby the original structures, algorithms can supply to the user which structures are in the immediate vicinity of the computed structure. This is of course of interest in many situations. For example, the input structure to StratiGraph can be the Kronecker structure delivered by the GUPTRI software. In this way the user may investigate whether any of the nearby structures are more likely from a physical point of view. The strata information can also be used in an enhanced staircase algorithm for computing the least (or the most) generic nearby Kronecker structure with a specific application in mind.

In this context, we will also continue our efforts to solve the following nearness problem (here illustrated for the matrix case). Given an $n \times n$ matrix A, the problem to determine the nearest matrix with a specified partial Jordan structure of size $p \times p$ ($p \leq n$) may be formulated as minimizing

$$f(Q, R) = \|AQ - QR\|_{\mathrm{E}} \tag{3}$$

over R and Q, where R is $p \times p$ with the prescribed partial Jordan structure and Q is $n \times p$ with orthonormal columns, i.e., Q belongs to the Stiefel manifold [14]. Lippert and Edelman [41] have developed Matlab tools that can be used to tackle this optimization problem.

The StratiGraph tool is under development and will be available via internet. Version 1.0 of StratiGraph [32] computes and presents qualitative results (strata information). The next generation of algorithms and tools to be developed within this project aim at producing quantitative results as well, including distance information concerning nearby structures [16].

Blocked QZ and GUPTRI for LAPACK. Recently, we have presented blocked algorithms for reduction of a regular matrix pair to generalized Schur form [7, 6]. The last step in the reduction is a blocked variant of the single-/double-shift QZ method applied to (A, B) in upper Hessenberg-triangular form. The blocking for improved data locality is obtained by restructuring the reference pattern of the updates associated with the bulge chasing in the QZ step. Timing results show that our new blocked variants outperform the current LAPACK routines, including drivers for the generalized eigenvalue problem, by a factor 2–5 for sufficiently large problems.

Our next step is to restructure the GUPTRI algorithm in terms of level 3 operations and increase the functionality regarding what (Jordan and Kronecker) structure information that should be computed. Different applications require different type of information and not necessarily the complete Jordan or Kronecker structure. For example, in the solution of a descriptor system we are mainly interested in the index of the system, which is the size of the largest Jordan block associated with the infinite eigenvalue. We propose to provide different options where also the placement of the structure information (blocks) can be controlled by the user.

Extracting Partial Canonical Structure for Large Scale Eigenvalue Problems. As outlined in the last project description one of our objectives is to investigate the possibility to compute partial canonical information of large scale eigenvalue problems. When A and B are very large and sparse matrices it may be prohibitively expensive to compute the complete generalized Schur decomposition of $A - \lambda B$ using standard methods for dense matrices.

Our method is a combination of the implicitly restarted Arnoldi (IRA) method with explicit restarts, a new Gerschgorin clustering heuristic and the staircase algorithm. It computes a partial Jordan-Schur form (or Weierstrass-Schur form in the pencil case) associated with the dominant eigenvalue [38]. Indeed, we compute an exact partial Jordan-Schur form of a regularized nearby problem $A + G$. The size of $\|G\|_F$ is a function of the deflation tolerance parameters used. A small value on $\|G\|_F$ indicates that A has a dominant eigenvalue with the computed Jordan structure.

The explicit restarts are used for extracting information about the Jordan blocks of the matrix that were not computed in previous applications of the IRA method. In theory each IRA application can only identify one Jordan block, but due to round-off errors we can accidentally get more information. However, in general it is not safe and reliable to rely on this property. In [38] we discuss what assumptions we make in order to get convergence and warrant meaningful and reliable results. These include the conditioning of the problems as well as the setting of dimension parameters (the IRA dimensions k, p, and the number of explicit restarts s) and deflation tolerances (tol and gap used for rank decisions).

The results from computational experiments conducted on problems with known canonical structure and varying ill-conditioning show that the methods deliver partial Jordan-Schur and Weierstrass-Schur forms of nearby problems with the correct canonical structure of the dominant eigenvalue. The distance from a sample problem to the corresponding regularized problem is proportional to the size of the deflation tolerance used. Due to the ill-posedness of the problem under consideration it is of course always possible to construct problems so that any of the three major building blocks produce erroneous result. Nonetheless, our study shows that it is possible to construct methods for extracting partial canonical structure of large scale eigenvalue problems.

Our future work include further investigations concerning alternative deflation criteria for the IRA method. Another direction is to investigate other methods for computing an eigenspace and associated projected problem. The most popular of these methods for eigenvalue problems are subspace iteration and the Lanczos, Arnoldi and rational Krylov methods (e.g., see [22, 28, 29, 45, 46, 47, 48]).

A Chordal-Based Definition for the Mean of a Cluster of Generalized Eigenvalues. Algorithms for extracting canonical information of matrices and matrix pairs approximate a multiple eigenvalue by the mean of a cluster of eigenvalues. For standard eigenvalue problems the mean of the eigenvalues is typically used to approximate the multiple eigenvalue (e.g., see [36]). The mean is an intuitive choice for this value but it also has the property that it is stable—if the problem is perturbed the change in the mean is bounded by the norm of the perturbation. Generalized eigenvalues are points on the complex projective plane. Our goal is to answer the question of how to define the mean of a cluster of generalized eigenvalues for this geometrically more complicated set. A cluster is identified from the computed eigenvalues by some heuristic.

The foundation of our definition is the chordal metric for generalized eigenvalue pairs (α_i, β_i). One interesting proposal is presented in [55, 53], where the mean is chosen as the point on the complex projective plane that minimizes the sum of squares of the chordal distances. It is shown that the mean is the unit eigenvector associated with the smallest eigenvalue of the 2×2 matrix $C^H D^{-1} C$, where $D = \mathrm{diag}(|\alpha_i|^2 + |\beta_i|^2)$ and C is a $2 \times n$ matrix with the β_is and the $-\alpha_i$s as its two columns. Computational experiments suggest that the mean is more stable than the individual eigenvalues. In particular, for matrix pairs with only one Weierstrass block the error in the mean is several orders smaller than the spectral variation. Our future work will include derivation of perturbation bounds for the mean and comparisons with other possible definitions.

International Collaboration. During several years the principal investigator have had a close collaboration with James Demmel, University of California, Berkeley on perturbation theory, applications, algorithms and software for singular matrix pencils (see references [10, 11, 12, 13]). Since 1994 we have an ongoing, fruitful collaboration with Alan Edelman, Massachusetts Institute of Technology (MIT) (see [15, 16]). Erik Elmroth visited MIT two weeks during Spring 1996. Alan Edelman has visited Umeå for two two-weeks periods (Spring 1994 and Summer 1996). Fall 1996 Erik Elmroth visited MIT for three months and Bo Kågström visited MIT for a week. We have also collaborated with Axel Ruhe, Chalmers University of Technology, Göteborg, Vera Kublanovskaya, Steklov Institute, St Petersburg, Gene Golub, Stanford University, and Paul Van Dooren, CESAME, Université Catholique de Louvain, Belgium on algorithms and applications for matrix pencils.

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