# PROBLEM SET 4

Math 505-Fall 2002

1. Let the scalar function $f(x)$ have a zero of multiplicity $p$ at $x = x^*$, i.e. $f^{(k)}(x^*) = 0,\ k = 0, 1, \ldots, p-1;\ f^{(p)} \neq 0$.

   (a) Show that Newton's method converges linearly (for sufficiently close $x^0$) with asymptotic convergence factor $(p-1)/p$.

   (b) If the Newton iterates are converging to this root of multiplicity $p$ show how to estimate $p$ from the computed quantities.

   (c) If the integer $p$ is known show that quadratic convergence can be restored by using the modified Newton's method:

$$x^{n+1} = x^n - p\frac{f(x^n)}{f'(x^n)} \ .$$

2. To compute the square root of $\alpha > 0$ Newton's method is applied to $x^2 - \alpha = 0$. This yields the sequence:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \ , \ n = 0, 1, \ldots \ .$$

   We consider three choices for the initial iterate $x_0$:
   (i) $x_0 = \alpha$, (ii) $x_0 = \beta + \alpha$, (iii) $x_0 = \gamma + \alpha\delta$.

   (a) Find the constants $\alpha, \beta, \gamma, \delta$ so that $x_0$ in each case best approximates $\sqrt{\alpha}$ in the interval $1/16 \leq \alpha \leq 1$ in the Chebyshev sense, i.e. so that

$$\max_{\frac{1}{16} \leq \alpha \leq 1} \left| x_0 - \sqrt{\alpha} \right| = \text{minimal} \ .$$

   (b) In each case estimate how many iterations it would take to produce an approximation to $\sqrt{\alpha}$ to full single precision in IEEE arithmetic. That is, so that

$$\frac{x_n - \sqrt{\alpha}}{\sqrt{\alpha}} \leq 2^{-24} \ , \ \text{for all } \alpha \in \left[\frac{1}{16}, 1\right] \ .$$

3. (a) With given points $\mathbf{x}^n$, $n = 0, 1, 2$ in the plane ($\mathbf{x} \in \mathbf{R}^2$) define $\mathbf{x}^{n+1}$ as the intersection of the planes $z = L_1(x, y)$ and $z = L_2(x, y)$ with $z = 0$ where $L_1(x, y)$ is the plane through the three points $(\mathbf{x}^i, z_1^i)$, $i = n, n-1, n-2$ and $z_1^i = f_1(\mathbf{x}^i)$. Similarly for $L_2(x, y)$ in terms of $z_2^i = f_2(\mathbf{x}^i)$. Find the formula for the components of $\mathbf{x}^{n+1}$.

(b) When will this iteration procedure fail to converge to a root of the system $(f_1(\mathbf{x}^i), f_2(\mathbf{x}^i))$? Is there a two-dimensional analog of the secant method that will always converge?

(c) Use this method to compute three iterates in atempting to solve:

$$(x + iy) = e^{(x+iy)} \ .$$

Take $\mathbf{x}_0 = (0.4, 1.4)$, $\mathbf{x}_1 = (0.2, 1.4)$, $\mathbf{x}_2 = (0.3, 1.1)$. Will this method always converge for this function?