Solution: Problem 3.5.1

Show that the nonlinear system

$$\begin{aligned}\dot{x} &= -y + xz^2\\ \dot{y} &= x + yz^2\\ \dot{z} &= -x(x^2+y^2)\end{aligned}$$

has a periodic orbit $\gamma(t) = (\cos t, \sin t, 0)^T$. Find the linearization of this system about $\gamma(t)$, the fundamental matrix $\Phi(t)$ for this (autonomous) linear system which satisfies $\Phi(0) = I$, and the characteristic exponents and multipliers of $\gamma(t)$. What are the dimensions of the stable, unstable and center manifolds of $\gamma(t)$?

Periodic Orbit: substitute into the system.

Derivative Matrix at Periodic Orbit:

$$Df = \begin{bmatrix} z^2 & -1 & 2xz \\ 1 & z^2 & 2yz \\ -2xz & -2yz & -(x^2+y^2) \end{bmatrix} \to Df\left(\gamma(t)\right) =: A(t) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Fundamental solution:

$$\Phi(t) = Ie^{At} = Q(t)e^{Bt}$$

so that the periodic matrix is the identity and the matrix exponent B is the linearization matrix A. As the spectrum of A (exponents) is $\pm i, -1$, it follows that the multipliers are $1, 1, e^{-2\pi}$.

Manifolds: We can see that the orbit $\gamma(t)$ is the intersection of the 2-dimensional center manifold, the xy-plane ($z = 0$) and the 2-dimensional stable manifold $x^2 + y^2 + z^2 = 1$. Rewrite system in polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$:

$$\begin{aligned}\dot{y} = x + yz^2 &\quad\to\quad \dot{\theta} = 1 \to \theta = t + \theta_0\\ \dot{x} = -y + xz^2 &\quad\to\quad \dot{r} = rz^2 \to z^2 + r^2 = C^2 = z_0^2 + r_0^2\\ \dot{z} = -x(x^2+y^2) &\quad\to\quad \dot{z} = -zr^2 \to \dot{r} = r(C^2 - r^2)\end{aligned}$$

which implies

$$r = \frac{Cr_0}{\sqrt{C^2 - r_0^2}}\left(A + e^{-2t}\right)^{-1/2}, \quad z = Ce^{-t}\left(A + e^{-2t}\right)^{-1/2}, A = \frac{r_0^2}{C^2 - r_0^2}$$

This shows that the stable manifold of $\gamma(t)$ is the unit sphere, and the orbits spiral onto the equator, i.e. $\gamma(t)$.