

DE Preliminary Exam

JANUARY 2004

Directions: Do all six of the following problems. Show all of your work, and justify all of your calculations.

1. Consider the ODE

$$\dot{x} = v \times x,$$

where $\times$ denotes the cross-product in $\mathbb{R}^3$. For each of the following scenarios, discuss the stability of $x = 0$:

- (a) $v \in \mathbb{R}^3$ is constant
- (b) $v = v(t) \in \mathbb{R}^3$, where $v(t + 2\pi) = v(t)$ for all $t \in \mathbb{R}$.

2. Consider the initial value problem

$$\dot{x} = (A + B(t))x, \quad x(0) = x_0,$$

where

$$A := \begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix}, \quad B(t) := \begin{pmatrix} e^{-t} & \operatorname{sech}(2t) \\ \frac{1}{1+t^2} & e^{-2t} \end{pmatrix}.$$

Show that $x(t) \rightarrow 0$ as $t \rightarrow +\infty$ for each initial value x_0 .

3. Consider the ODE

$$\begin{aligned} \dot{x} &= x - y - x^3 \\ \dot{y} &= x + y - y^3. \end{aligned}$$

Show that:

- (a) there is at least one nontrivial periodic solution
- (b) the periodic solution found in part (a) is unique and globally attracting.

4. Consider the IVP

$$u_t - (xu)_x = 0; \quad u(x, 0) = u_0(x),$$

where $(x, t) \in \mathbb{R} \times \mathbb{R}^+$.

- (a) Find the characteristics through $(x, t) = (x_0, 0)$ and sketch them.
- (b) Discuss the behavior of the solution along the characteristic through $(x, t) = (x_0, 0)$.
- (c) Find $u(x, t)$.

5. Consider the IBVP

$$u_t - u_{xx} = 0; \quad u(x, 0) = f(x), \quad u(0, t) = u(\pi, t) = 0.$$

where $(x, t) \in (0, \pi) \times \mathbb{R}^+$.

(a) Find a candidate for the solution of the IBVP of the form

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \phi_n(x), \tag{0.1}$$

where $\{\phi_n(x)\}_{n=1}^{\infty}$ is a complete set of eigenfunctions.

(b) Give sufficient conditions on f such that

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N f_n \phi_n(x) = f(x),$$

where f_n is appropriately defined for each n .

(c) Outline a proof that equation (0.1) actually solves the IBVP using uniform convergence arguments. State any and all assumptions on f that you require.

(d) State a maximum principle for the IBVP, and show that $b_1(t)\phi_1(x)$ satisfies it.

6. Solve the IVP for the nonhomogeneous Klein-Gordon equation (KGE),

$$E_{xx} - \kappa^2 E - E_{tt} = S(x, t); \quad E(x, 0) = E_t(x, 0) = 0,$$

as follows:

(a) Apply Duhamel's principle; that is, find the IVP satisfied by $e(x, t; s)$ so that $E(x, t) = \int_0^t e(x, t; s) ds$. (Hint: e should satisfy a homogeneous 1D KGE.)

(b) Define $f(x, y, t; s) := e^{i\kappa y} e(x, t; s)$, and show that f satisfies a homogeneous 2D wave equation. Solve for f using the fact that the solution $u(x, t)$, $x \in \mathbb{R}^2$, of the IVP

$$u_{tt} = \Delta u; \quad u(x, 0) = h(x), \quad u_t(x, 0) = g(x)$$

is

$$u(x, t) = \frac{1}{2\pi} \frac{d}{dt} \iint_{|\xi| < t} \frac{h(x + \xi)}{t^2 + |\xi|^2} d\xi + \frac{1}{2\pi} \iint_{|\xi| < t} \frac{g(x + \xi)}{\sqrt{t^2 + |\xi|^2}} d\xi.$$

(c) Show that

$$e(x, t; s) = -\frac{1}{2} \int_{-(t-s)}^{t-s} J(\kappa \sqrt{(t-s)^2 - \xi_1^2}) S(x + \xi_1, s) d\xi_1,$$

where

$$J(x) := \int_{-1}^1 \frac{e^{ixs}}{\sqrt{1-s^2}} ds.$$

(d) Find $u(x, t)$.