# ODE/PDE Spring 2003 Qualifying Exam

*Instruction:* Complete all four problems. Clear and concise answers will improve your score.

1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ denote a $C^1$ function and consider the system

$$\frac{du}{dt} = f(u)$$

for a vector function $u = u(t)$.

a) What is a fixed point (in other terminology: critical point) of the system?

b) When is a fixed point called stable in the sense of Lyapunov?

c) Let $u^*$ be a fixed point. How can one obtain stability or instability (in the sense of Lyapunov) using the eigenvalues of a certain matrix? (You do not have to prove the criteria.)

d) Given the system

$$\begin{aligned} u_1' &= u_1(1 - u_2) \\ u_2' &= u_2(4 - u_1) \end{aligned}$$

Determine all fixed points and discuss their stability.

2. a) Solve the scalar initial value problem

$$\begin{aligned} u'(t) &= |u(t)|, \quad t \geq 0 , \\ u(0) &= a , \end{aligned}$$

explicitly. Is the solution $u = u(t, a)$ everywhere differentiable as a function of $a \in \mathbb{R}$?

b) Consider a scalar initial value problem

$$\begin{aligned} u'(t) &= f(u(t), t), \quad t \geq 0 , \\ u(0) &= a , \end{aligned}$$

where $f : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is a given function. State conditions (without proof) which guarantee that the problem has a unique solution $u = u(t, a)$ existing for all time $t \geq 0$. Give conditions so that the partial derivative

$$u_a(t, a) = \frac{\partial u(t, a)}{\partial a}$$

exists and is continuous. Then prove the derivative is positive.

3. If $u = u(t) = u(x,t)$ then an initial boundary value problem (IBVP) for $u$ is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \tag{1}$$

$$u(x,0) = f(x)\,, \quad u(0,t) = 0\,, \quad u(1,t) = 0\,, \tag{2}$$

where $0 \le x \le 1$ and $t \ge 0$ and $f = f(x)$ is a given continuous function on $[0,1]$.

(a) Use separation of variables and Fourier series to give a formula for the solution $u$ of the IBVP (1) and (2) in terms of the initial data $f$.

(b) Is (1) elliptic, parabolic or hyperbolic? Why? If there is a maximum principle for this class of equations, state it. If this equation has characteristics, what are they? If $f(x) \equiv \sin(2\pi x)$, where is the maximum value of $u(x,t)$ for $0 \le x \le 1$ and $t \ge 0$?

(c) Prove that the solution is infinitely differentiable for $t > 0$.

(d) If $f(x) \equiv 1$, is the Fourier series solution solution continuous at $(x,t) = (0,0)$?

(e) If

$$\int_0^1 f(x) \sin(\pi x)\, dx = 0$$

what does the solution $u(x,t)$ look like for moderately large times? Give a formula and a sketch.

(f) The solution of the IBVP can be written in the form

$$u(x,t) = \int_0^1 G(x,y,t) f(y)\, dy\,,$$

where $G$ is the Green's function for the IBVP. Give a formula for $G$.

(g) $F = F(x,t)$ is a continuous function on $t \ge 0$, $0 \le x \le 1$, use Duhamel's principle to solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F$$

along with the boundary conditions (2). Use the Green's function.

4. If $u = u(t) = u(x,t)$ then an initial value problem (IVP) for $u$ is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \tag{3}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{4}$$

where $t \geq 0$, $-\infty < x < \infty$, $u_t = \partial u/\partial t$ and $f = f(x)$ and $g = g(x)$ are a given continuous functions with compact support.

(a) Derive d'Alembert's solution to (3) by first changing the equation to the variables $\xi = x + t$, $\eta = x - t$ and verifying that the resulting equation has solutions of the form $A(\xi) + B(\eta)$. Transform these solutions back to the $x, t$ coordinates and then find a solution that satisfies the initial conditions.

(b) Is (3) elliptic, parabolic or hyperbolic? Why? If there is a maximum principle for this class of equations, state it. If this equation has characteristics, what are they?

(c) If

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } a \leq x \leq b \\ 0 & \text{otherwise}, \end{cases}$$

is the characteristic function of the interval $[a,b]$, and if $f(x) = \chi_{[-1,1]}(x)$ and $g(x) \equiv 0$, then what is d'Alembert's solution to the IVP? Make a sketch in the $(x,t)$-plane of where this solution is non-zero. Is this solution continuous? Differentiable?

(d) State reasonable conditions on $f$ and $g$ so that d'Alembert's solution to (3) and (4) is a *classical* solution. Is the solution with $f(x) = \chi_{[-1,1]}(x)$ and $g(x) \equiv 0$ a classical solution?

(e) The energy $E = E(t)$ for the wave equation is the sum of the kinetic and potential energies:

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} \left( u_t^2(x,t) + u_x^2(x,t) \right) \, dx \, .$$

Show that $E(t)$ is constant.

(f) Show that the IVP (3) and (4) is well posed.

(g) The solution of the IBVP can be written in the form

$$u(x,t) = \int_0^1 G(x,y,t) f(y) \, dy + \int_0^1 H(x,y,t) g(y) \, dy \, ,$$

where $G$ and $H$ are Green's functions for the IBVP. Give formulas for $G$ and $H$. How are $G$ and $H$ related?