ODE Exam
January 2000

Instructions: Answer one of problems 1 and 2, and one of problems 3 and 4. Circle the two problems you want graded.

1. Consider the nonlinear oscillator (NO) $\ddot{x} + g(x) = 0$ or in system form $\dot{x} = y, \dot{y} = -g(x)$.

   (a) Define a critical point (CP) (x_c, y_c) in terms of g.

   (b) Let (x_c, y_c) be a CP. Linearize the NO equation about the CP and discuss linearized stability in terms of g. What can you conclude at this stage about (nonlinear) stability?

   (c) Show that $E(x, y) = \frac{1}{2}y^2 + G(x)$ where $G'(x) = g(x)$ is constant along any solution of the NO equation.

   (d) Construct the phase plane portrait for the NO with $g(x) = x - \frac{1}{a-x}$ for $x < a$. Discuss the qualitative behavior of solutions for all IC's (x_0, y_0) and $x_0 < a$. Include the CPs and their (nonlinear) stability in your discussion.

2. Consider $\dot{x} = Ax \quad (*)$, where A is a constant $n \times n$ matrix.

   (a) Define e^{At} and show that the solution of the IVP $x(0) = x_0$ is $x(t) = e^{At}x_0$.

   (b) Show that $e^{A(t+\tau)} = e^{At}e^{A\tau}$ by using a uniqueness theorem for solutions of $*$.

   (c) Find e^{At} for $A = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$. Verify that $\left(e^{A(t+\tau)}\right)_{11} = \left(e^{At}e^{A\tau}\right)_{11}$ from your explicit formula.

   (d) Find the solution of $\dot{x} = Ax + f(t)$ in terms of e^{At} and $f(t)$.

3. Consider the IVP $\dot{x} = f(x,t), x(0) = z$, and let $\varphi(t,z)$ be the solution with $\varphi(0,z) = z$.

(a) Define continuity of φ in z for fixed t.

(b) Specify general conditions on f so that solutions of the IVP exist uniquely on some open interval containing $t = 0$ and are continuous in z for fixed t.

(c) Prove the continuity in z for fixed t for your conditions in (b). Use a form of the Gronwall inequality in your proof.

(d) Prove the version of the Gronwall inequality you used in (c).

4. Suppose $x_p(t)$ is a T-periodic solution of $\dot{x} = f(x), x \in \mathbf{R}^n$.

(a) Define u by $x = x_p + u$ and find the linearized equation for u, i.e., find $A(t)$ in terms of f such that the linearized equation is $\dot{u} = A(t)u$ $(*)$ where $A(t+T) = A(t)$.

(b) Let $\Phi(t)$ be the PSM for $*$, i.e., $\dot{\Phi} = A(t)\Phi, \Phi(0) = I$. Prove that $\Phi(t+T) = \Phi(t)\Phi(T)$.

(c) Let B be a matrix such that $\Phi(T) = e^{BT}$. Show that $P(t)$ defined by $\Phi(t) = P(t)e^{Bt}$ is T-periodic, i.e., $P(t+T) = P(t)$.

(d) Discuss the linearized stability of $x_p(t)$ in terms of the result in (c).