Original

ODE & PDE exam–August 2005

ID#:____

August 16, 2005

WORK ALL FOUR PROBLEMS

ODE PART

1. (25 pts) Consider the nonlinearly damped mass-spring system

$$m\frac{d^2x}{dt^2} + f(x, \frac{dx}{dt})\frac{dx}{dt} + \lambda x = 0 \tag{1}$$

with $\lambda, m > 0$ and smooth $f(x, x') \geq 0$ in a neighborhood of the origin in the (x, y) phase plane (i.e. with $y = dx/dt$).

(a) (15pts) Show that the non-negative function

$$V(x, y) = \frac{1}{2}\left(\lambda x^2 + my^2\right) \tag{2}$$

is a weak Lyapounov function for equation (1)

(b) (5 pts) Use the result in part (1) to show that the zero solution is stable.

(c) (5 pts) Is the zero solution always asymptotically stable?

2. (25 pts) Consider the system

$$\mathbf{x}' = P(t)\mathbf{x} \; , \; P \in R^n \; , \; P(t+T) = P(t) \; , \tag{3}$$

with $P(t)$ continuous and real $T > 0$.

(a) (5pts) State (without proof) the Floquet theorem for this system, and discuss its implications for the existence of periodic solutions.

(b) (10 pts) Given the 2×2 system

$$\frac{d\mathbf{x}}{dt} = P(t)\mathbf{x} \; , \; \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \; , \; P(t) = \begin{pmatrix} -\sin 2t & \cos 2t - 1 \\ \cos 2t + 1 & \sin 2t \end{pmatrix} \tag{4}$$

Show that a Fundamental matrix is given by

$$\Phi(t) = \begin{pmatrix} e^t(\cos t + \sin t) & e^{-t}(\cos t + \sin t) \\ e^t(\cos t - \sin t) & -e^{-t}(\cos t - \sin t) \end{pmatrix} \tag{5}$$

(c) (5 pts) Find the characteristic numbers $\mu_i \; , \; i = 1, 2$ for the system in part b, so that there exist solutions of the form (with T the period of P):

$$\mathbf{y}_i(t+T) = \mu_i \mathbf{y}_i(t) \; , \; i = 1, 2. \tag{6}$$

(d) (5 pts) Does the system in (part b) have a periodic solution? If yes, what is its period? If no, why not?

PDE PART

1. (25 pts) Solve the initial value problem

$$u_t + u_x = u^2, \quad u(x,0) = \frac{1}{2}\cos x \; , \tag{7}$$

and determine the largest time $T > 0$ for which $u(x,t)$ is finite for

$$-T < t < T, \quad -\infty < x < \infty \; . \tag{8}$$

2. (25 pts) Solve the initial value problem

$$u_{tt} = u_{xx} - 3u, \quad u(x,0) = \cos x, \quad u_t(x,0) = \sin x \; . \tag{9}$$