ODE & PDE exam–January 2006

January 9, 2006

ODE PART: WORK BOTH PROBLEMS

1. (25 pts)

   Consider the system

$$\begin{aligned} \frac{dx}{dt} &= ay + \left(x^2 - y^2\right) \\ \frac{dy}{dt} &= -ax - 2xy \ . \end{aligned} \tag{1}$$

   (a) (3pts) Show that the function

$$H(x,y) = \frac{a}{2}\left(x^2 + y^2\right) + \left(x^2 y - \frac{1}{3}y^3\right)$$

   is a constant of the motion.

   (b) (4 pts) Find all equilibria.

   (c) For $a \neq 0$:

      i. (5pts) Derive the linearized system valid in the neighborhood of (x_0, y_0), where (x_0, y_0) is an arbitrary equilibrium point of eq.(1).

      ii. (4pts) Give the type and linear stability of each critical point of eq.(1).

      iii. (5pts) Discuss the nonlinear stability of each critical point in part c(ii) and sketch the phase plane portrait.

   (d) (4pts) If $a = 0$ discuss the equilibrium point $(0, 0)$ and its stability and sketch the phase plane portrait.
   (*Hint:* you may consider solutions of the form $y = kx$).

2. (25 pts)

Consider the system

$$\begin{aligned} \frac{dx}{dt} &= x - y - 2x^3 - xy^2 \\ \frac{dy}{dt} &= x + y - 2y^3 - x^2 y \ . \end{aligned} \tag{2}$$

(a) (5pts) Show that the critical point at the origin is unstable (you can assume, without proof, that there are no other critical points).

(b) (10 pts) Show that there is at least one limit cycle.

(c) (10 pts) Show that there exists a unique, globally attracting limit cycle.

PDE PART: WORK BOTH PROBLEMS

1. (25 pts) Consider the IVP for the 3D wave equation with spherical symmetry:

$$u_{xx} + u_{yy} + u_{zz} = \frac{1}{c^2} u_{tt} \ , \ u(x, y, z, 0) = 0 \ , \ u_t(x, y, z, 0) = h(r) \ , \tag{3}$$

where $r := (x^2 + y^2 + z^2)^{1/2}$ and $h(r)$ is square integrable in $\mathbf{R}^3$.

(a) (5pts) Find the IVP satisfied by $u(r, t)$.

(b) (5pts) Show that $\Psi(r, t) := ru(r, t)$ satisfies the 1D wave equation

$$\Psi_{rr} = \frac{1}{c^2} \Psi_{tt} \ , \ r \geq 0 \ . \tag{4}$$

with appropriate conditions at $r = 0$.

(c) (15pts) Find the solution to the IVP in part (a) for u by solving the associated IBVP eq.(4) for $\Psi(r, t)$.

2. (25 pts)

Consider the following BVP for $u(x, y, t)$:

$$\begin{aligned} u_{xx} + u_{yy} - \frac{1}{c^2} u_{tt} &= e^{-i\omega t} \delta(x) \delta(y - y_0) \ , \\ u(x, 0, t) = 0 &= u(x, \pi, t) \ . \end{aligned} \tag{5}$$

Here, $x \in \mathbf{R}$, $y, \ y_0 \in (0, \pi)$ and $\omega > 0$.

(a) (5pts) Solve the eigenproblem

$$-\psi_{yy} = \lambda \psi \ , \ \psi(0) = 0 = \psi(\pi) \ . \tag{6}$$

State the relevant completeness theorem and find the eigen-representation for $\delta(y - y_0)$.

(b) (10pts) Look for a solution of eq.(5) of the form

$$u(x, y, t) = \sum_{n=1}^{\infty} G_n(x) e^{-i\omega t} \sin ny \ . \tag{7}$$

Show that G_n must satisfy an equation of the form

$$\frac{d^2 G_n}{dx^2} + k_n^2 G_n = c_n \delta(x) \tag{8}$$

and find k_n^2 and c_n. Determine n such that $k_n^2 \geq 0$ or $k_n^2 < 0$.

(c) (10 pts) Solve eq.(8) for the Green's function $G_n(x)$ under the condition that G_n be bounded. Consider two cases. For $k_n^2 > 0$ find the unique Green's function giving traveling waves that are outgoing at infinity (i.e. traveling to the right as $x \to \infty$ and to the left as $x \to -\infty$. For $k_n^2 < 0$, discuss the qualitative behavior of the Green's function.