ODE Exam, Winter 1999

1. Consider $x'' + x^3 + x^2 - 2x = 0$.
a. Find the critical points (equilibrium solutions).
b. Linearize about the critical points and solve the linearized equations. Discuss the stability of the critical points based on the linearization. If the critical point is a center find the period of the linearized equation.
c. Construct the energy function and use it to sketch the phase plane portrait. Discuss the stability of the critical points based on your result.
d. For what initial conditions $x(0) = x_0, x'(0) = y_0$ are the solutions of the ode periodic. What happens to the period as a separatrix is approached.

2. Consider $\mathbf{x}' = A(t)\mathbf{x}$, where

$$A(t) = \begin{bmatrix} -1 + (3/2)\cos^2 t & 1 - (3/4)\sin 2t \\ -1 - (3/4)\sin 2t & -1 + (3/2)\sin^2 t \end{bmatrix}.$$

a. Verify that the vector $(-\cos t, \sin t)e^{t/2}$ is a solution of the linear system.
b. Find the Floquet(characteristic) multipliers associated with the period 2π of A. We do not recommend trying to find a second linearly independent solution.

3. Let $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a C^1 function with $|f_x| \leq 1$ for all $(t, x) \in \mathbf{R} \times \mathbf{R}$. Let $a(\lambda) \in C^1(\mathbf{R})$ satisfy $|a'(\lambda)| \leq 1$ for all λ. Denote the solution of the initial value problem

$$x' = f(t, x), \quad x(0) = a(\lambda)$$

by $x_\lambda(t)$.
a. Show that for t in the domain of both $x_\lambda(t)$ and $x_\gamma(t)$

$$|x_\lambda(t) - x_\gamma(t)| \leq |\lambda - \gamma| + \int_0^t |x_\lambda(s) - x_\gamma(s)|\, ds.$$

b. Show that whenever it exists,

$$\left|\frac{\partial}{\partial \lambda} x_\lambda(1)\right| \leq 3.$$