August 1994 ODE/PDE EXAM
ID NUMBER:

Part 1. Solve two of the four problems given below.
Ph.D students: At most one question can be of the M.A. type.
Only two problems will be graded. Write your problem numbers here:

1. (M.A.) Consider the system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ with $\mathbf{x}(t),\ \ \mathbf{f}(t)$ in $\mathbf{R}^n$, and $\mathbf{f}(t) = \mathbf{f}(t+T)$. All of the eigenvalues of the constant $n \times n$ matrix A satisfy the constraint, $\mathcal{R}e(\lambda) < 0$.

a. Write down a formula for a general solution.

b. Write down a formula for a T-periodic solution $\mathbf{h}(t)$ and show that it is unique.

c. Show that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{h}(t)$.

2. (M.A.) Let A be an arbitrary 2×2 matrix with real entries. Then (by Cayley-Hamilton) A satisfies its characteristic equation, i.e., if $p(\lambda) = det(A - \lambda I)$, then $p(A) = 0$.

a. Show that $Le^{tA} = 0$, where the differential operator L is defined by $L = p(\frac{d}{dt})$.

b. Assume that the following is valid. (The functions $v(t)$ and $w(t)$ are real-valued.)

$$\begin{aligned} e^{tA} &= v(t)I + w(t)A, \\ \frac{d}{dt}e^{tA} &= \frac{dv}{dt}(t)I + \frac{dw}{dt}(t)A. \end{aligned}$$

Show that $Lv(t) = 0$ and $Lw(t) = 0$. Find $v(0)$, $w(0)$ $v'(0)$ $w'(0)$.

c. Use this information to construct e^{tA} where

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}.$$

3. Show that

$$\begin{aligned}\frac{dx}{dt} &= F(x,y) = -2x - 3y - xy^2,\\ \frac{dy}{dt} &= G(x,y) = y + x^2 - x^2y,\end{aligned}$$

has no periodic solutions that are not constant.
Hint: Use Green's Theorem to study

$$\oint_C [F\,dy - G\,dx]$$

on a smooth closed curve C.

4.

a. Show that

$$\mathcal{H} = \frac{1}{2}y^2 + \frac{1}{2}v^2 - \frac{1}{2}\lambda^2 x^2 - \frac{1}{2}\mu^2 u^2 + \frac{1}{2}ux^2$$

is a Hamiltonian for

$$\begin{aligned}\frac{dx}{dt} &= y, \quad \frac{dy}{dt} = \lambda^2 x - ux,\\ \frac{du}{dt} &= v, \quad \frac{dv}{dt} = \mu^2 u - \frac{1}{2}x^2.\end{aligned}$$

b. Show that for $\lambda^2 = \mu^2$, the above system can be reduced to a two-dimension system. {Hint: Set $u = \alpha x,\ \ v = \alpha y$ with a suitable α.}

c. Find a Hamiltonian for the reduced system.

d. Discuss the critical points and sketch a phase plane portrait of the reduced system.

August 1994 ODE/PDE EXAM

Part 2. Solve two of the four problems given below.
Ph.D students: At most one question can be of the M.A. type.
Only two problems will be graded. Write your problem numbers here:

1. (M.A.)

a. Classify the PDE,

$$u_{xx} + \alpha u_{yy} + \beta u_y = \lambda^2 u,$$

for all values of α and β.

b Solve for $u(x, y)$ with $\alpha = \beta = \lambda^2 = 0$ on $0 < x < 1$.
Is there a maximum principle in this case? Justify your answer with the solution you obtained.

c Set $\beta = 0$ and $\alpha = -c^2$. Solve for

$$u = f(s) = \sum_{n=0}^{\infty} a_n s^n,$$

where $s = y^2 - c^2x^2$. (First find an equation for $f(y^2 - c^2x^2$.)

2. (M.A.) State and prove a theorem regarding the attainment of maxima and minima by solutions to the I.B.V.P.

$$\begin{aligned} u_t &= u_{xx}, \;\; 0 < x < L, \;\; t > 0, \\ u(t, 0) &= u(t, L) = 0, \; t > 0, \\ u(0, x) &= f(x) \varepsilon C^2(0, L). \end{aligned}$$

3.

a. Show that

$$u(x, y) = \frac{1 - x^2 - y^2}{1 - 2x + x^2 + y^2}$$

is harmonic for $x^2 + y^2 < 1$.

b. Is it harmonic for $x^2 + y^2 < a$ if $a > 1$? Explain.

c. Prove that

$$\frac{a-r}{a+r} \leq u(x,y) \leq \frac{a+r}{a-r}$$

for $r < a$ is true for all $a < 1$, where $r = \sqrt{x^2+y^2}$.

d. Prove that

$$\frac{1-a}{1+a} \leq u(x,y) \leq \frac{1+a}{1-a}$$

for $\sqrt{x^2+y^2} \leq a$ is true for all $a < 1$.

4. Consider the following equation that arises in the study of elasticity,

$$\rho \mathbf{u}_{tt} = (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u}) + \mathbf{F}(\mathbf{x}); \quad \mathbf{u} = (u_1, u_2, u_3)$$

in $\Omega \subset \mathbf{R}^3$ with $\mathbf{u} = \mathbf{0}$ on the boundary of Ω. (ρ, λ, μ are constants.)

a. Show that if $\mathbf{F} = 0$, then

$$E(t) = \int_\Omega \frac{1}{2}[\rho|\mathbf{u}_t|^2 + (\lambda+\mu)(\nabla \cdot \mathbf{u})^2 + \mu \sum_{i=1}^{3} |\nabla u_i|^2] d\mathbf{x}$$

is a constant.

b. Assume $\mathbf{F} \neq \mathbf{0}$, and set $\mathbf{F} = \nabla U + \nabla \times \theta$, and $\mathbf{u} = \nabla \phi + \nabla \times \mathbf{A}$. Find the equations for ϕ and $\mathbf{A}$.

c. Taking $\phi = \phi_0$, $\mathbf{A} = \mathbf{A}_0$ (constants) on $\partial\Omega$, find the Lagrangian for the $(\phi, \mathbf{A})$ equations in b.