

ODE/PDE Qualifying Examination

August 1999

SS #: _____

Directions:

Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages and write your SS # in each page. Indicate clearly below which questions you are attempting.

Questions:

ODE part: Answer *any two* of problems 1-3 and *one* of problems 4-5.

1. (a) Solve $x' = 2\frac{x^2}{t^2}$, with initial condition $x(1) = 1$. Find the maximum time $T > 1$, so that the solution is defined on $[1, T)$.
- (b) Find all the solutions of $x' = 2\frac{x^2}{t^2}$, with initial condition $x(0) = 0$. Does your answer contradict the uniqueness theorem? Explain.
2. Consider the nonlinear ODE system

$$\begin{aligned}x' &= x - 1 - y \\ y' &= -y - (x - 1)^3\end{aligned}\tag{1}$$

- (a) Show that (1) has only one equilibrium point $\bar{x}_* = (x_*, y_*)$. Find the linearized equations around $\bar{x}_* = (x_*, y_*)$ and discuss the nonlinear stability of the equilibrium point.
- (b) Write the linearized equations in matrix form, $\bar{x}' = A\bar{x}$, and find the exponential e^{At} of the matrix A .
3. Consider the nonlinear second order equation

$$x'' = -x + x^2\tag{2}$$

- (a) Find the total energy $E = E(x, x')$ for this equation and show that E is conserved.
- (b) Use the total energy E to sketch the solutions of (2) in the phase plane (x, x') .
- (c) Identify the equilibrium points of (2) and determine whether they are stable or unstable.
- (d) For what values of A is the solution of (2) with initial conditions $x(0) = A$, $x'(0) = 0$ periodic?
4. (a) Prove Gronwall's inequality: If $u(t) \geq 0$ is a continuous function on $0 \leq t \leq T$ and satisfies

$$u(t) \leq u_0 + \int_0^t K(s)u(s) ds, \quad 0 \leq t \leq T$$

with $u_0 \geq 0$ and $K(t)$ a non-negative continuous function on $0 \leq t \leq T$, then $u(t)$ satisfies

$$u(t) \leq u_0 e^{\int_0^t K(s) ds}, \quad 0 \leq t \leq T.$$

- (b) Use Gronwall's inequality to show that there is a unique solution $u(t)$ of the initial value problem

$$\begin{aligned}u' &= t \sin(u + t), \quad 0 < t < \infty \\u(0) &= 1\end{aligned}\tag{3}$$

You can assume that any solution $u(t)$ of (3) is defined on $0 \leq t < \infty$.

5. Consider the nonlinear ODE system

$$\begin{aligned}x' &= -x - y + x(x^2 + y^2), \quad x(0) = x_0, \\y' &= x - y + y(x^2 + y^2), \quad y(0) = y_0.\end{aligned}\tag{4}$$

- (a) Show that $V(x, y) = x^2 + y^2$ is a Liapunov function in a neighborhood of $(x, y) = (0, 0)$.
(b) Utilize $V(x, y)$ to prove directly (that is, without quoting Liapunov's theorem) that $(0, 0)$ is an asymptotically stable equilibrium point.
(c) Find $R > 0$ so that if the initial data (x_0, y_0) is in the disc of center $(0, 0)$ and radius R , then the solution $(x(t), y(t))$ of (4) converges to $(0, 0)$ as $t \rightarrow \infty$.

PDE part: Answer *any two* of problems 1-3 and *one* of problems 4-5.

1. Consider the initial-boundary-value problem

$$\begin{aligned}u_t &= u_{xx} + e^{-t} \sin \pi x - \sin 3\pi x \\u(0, t) &= 0 \\u(1, t) &= 0 \\u(x, 0) &= \sin \pi x\end{aligned}$$

Find the solution and discuss its behavior as $t \rightarrow \infty$.

2. Solve Laplace's equation on the disk, that is, solve the boundary-value problem

$$\begin{aligned}u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad 0 < r < 1, \quad 0 \leq \theta \leq 2\pi, \\u(1, \theta) &= f(\theta), \quad 0 \leq \theta \leq 2\pi, \\u(r, \theta) &\text{ bounded around } r = 0.\end{aligned}$$

3. (a) Solve the initial-value problem

$$\begin{aligned}u_{tt} - u_{xx} &= 0, \quad -\infty < x < \infty, \quad 0 < t < \infty \\u(x, 0) &= \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \\u_t(x, 0) &= 0\end{aligned}$$

- (b) For fixed $t > 0$, identify the x -region where $u = 0$ and where $u \neq 0$.

4. Consider the initial-value problem

$$\begin{aligned}\rho_t - (y\rho)_y &= \rho_t - y\rho_y - \rho = 0, & -\infty < y < \infty, & \quad 0 < t < \infty, \\ \rho(y, 0) &= \rho_0(y).\end{aligned}$$

- (a) Solve the initial-value problem and sketch the characteristics in the y - t plane.
(b) Use your solution in a) with the initial data

$$\rho_0(y) = \begin{cases} 1 & y \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

to compute $\int_{-\infty}^{\infty} \rho(y, t) dy$.

- (c) Can you verify your result in b) directly from the PDE? Explain.

5. Consider the boundary-value problem

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad 0 < y < \infty \quad (1)$$

$$u(x, 0) = f(x) \quad (2)$$

and its solution

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x - \xi)^2 + y^2} f(\xi) d\xi \quad (3)$$

- (a) Show that $u(x, y)$ in (3) satisfies (1).
(b) Plug $y = 0$ into the integrand in (3). Does it satisfy the boundary condition in (2)? Explain. Your explanation could be in terms of delta sequences.
(c) Find $u(x, y)$ explicitly for $f(x)$ given by

$$f(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

and prove that the boundary condition in (2) is satisfied.