PDEs, Jan. 2000

Solve two out of the following three problems.

1) If (x, y) denote Cartesian coordinates in the plane, then the Laplace operator is, by definition,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \ .$$

Formulate, as precisely as you can, what it means that the Laplace operator in polar coordinates (r, θ) reads

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

and prove this statement.

sol'n dist

2) Consider the equation

$$u_t(x,t) = x u_x(x,t), \quad x \in \mathbb{R}, \quad t \geq 0 \ ,$$

with initial condition

$$u(x,0) = \sin x \ .$$

a) Sketch the characteristics and solve the problem.

b) Solve the equation

$$u_t(x,t) = x u_x(x,t) + 1$$

with the same initial condition.

3) Consider the 1D wave equation

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), \quad c > 0 \ ,$$

with initial condition

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in \mathbb{R} \ ,$$

where f and g are real C^2 functions.

a) Solve the problem for

$$f(x) \equiv 1, \quad g(x) = 2\cos x, \quad c = 2 \ .$$

b) Assume that the solution $u(x,t)$ of the general problem is 2π-periodic in x. Show that the 'energy'

$$\|u_x(\cdot,t)\|^2 + \frac{1}{c^2}\,\|u_t(\cdot,t)\|^2$$

is constant in time. Here

$$\|v(\cdot)\|^2 = \int_0^{2\pi} v^2(x)dx \,.$$