Van der Pol without & with forcing, Forced Duffing

<u>Holmes Notation</u>

$$\ddot{x} + \varepsilon\alpha(x^2-1)\dot{x} + x = \varepsilon\beta\cos\omega t$$

system Form

$$\dot{x} = y - \varepsilon\alpha\left(\tfrac{1}{3}x^3 - x\right)$$

$$\dot{y} = -x + \varepsilon\beta\cos\omega t$$

<u>Case 1</u> $\quad \beta = 0 \quad , \quad 0 < \varepsilon \ll 1$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ -\sin t & -\cos t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} +\cos t & -\sin t \\ -\sin t & -\cos t \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \varepsilon\alpha\left\{\tfrac{1}{3}(u\cos t - v\sin t)^3 - (u\cos t - v\sin t)\right\}\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

Averaged system

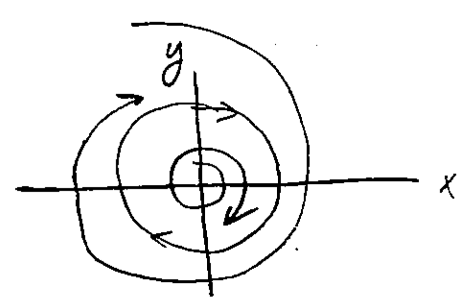
$$\begin{pmatrix} \dot{\bar{u}} \\ \dot{\bar{v}} \end{pmatrix} = \varepsilon\alpha\,\tfrac{1}{2}\left[1 - (\bar{u}^2+\bar{v}^2)/4\right]\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

Hopefully
$$u = \bar{u} + O(\varepsilon)$$
$$v = \bar{v} + O(\varepsilon) \quad \text{for} \quad 0 \le t \le T/\varepsilon$$

$$x = \underbrace{\bar{u}}_{=r(t)\cos\varphi(t)}\cos t - \underbrace{\bar{v}}_{r(t)\sin\varphi(t)}\sin t + O(\varepsilon)$$
$$y = -\bar{u}\sin t - \bar{v}\cos t + O(\varepsilon)$$

$$\Rightarrow \quad x(t) = \bar{r}(t)\cos(t + \bar{\varphi}(t)) + O(\varepsilon)$$
$$y(t) = -r(t)\sin(t + \varphi(t)) + O(\varepsilon)$$

$$\dot{\bar{r}} = \varepsilon\alpha\frac{\bar{r}}{2}\left(1 - \frac{\bar{r}^2}{4}\right) \quad \Rightarrow$$
$$\dot{\bar{\varphi}} = 0$$



Case 2 $\quad \alpha$ and $\beta \neq 0 \quad 0 < \varepsilon \ll 1$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos \omega t & -\frac{1}{\omega} \sin \omega t \\ -S & -\frac{1}{\omega} C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \frac{1-\omega^2}{\omega} x \begin{pmatrix} S \\ C \end{pmatrix} + \varepsilon \alpha \left(\tfrac{1}{3} x^3 - x\right) \begin{pmatrix} -C \\ S \end{pmatrix} - \varepsilon \beta \frac{1}{\omega} \cos \omega t \begin{pmatrix} S \\ C \end{pmatrix}$$

$$\sigma = \frac{1-\omega^2}{\varepsilon \alpha} \quad \text{and averaging} \Rightarrow$$

$$\dot{\bar{u}} = \frac{\varepsilon \alpha}{2\omega} \left[\bar{u} - \sigma \bar{v} - \frac{\bar{u}}{4} \left(\bar{u}^2 + \bar{v}^2 \right) \right]$$

$$\dot{\bar{v}} = \frac{\varepsilon \alpha}{2\omega} \left[\sigma \bar{u} + \bar{v} - \frac{\bar{v}}{4} \left(\bar{u}^2 + \bar{v}^2 \right) \right] - \frac{\varepsilon \beta}{2\omega} \qquad *$$

For $\omega = 1$ and $\beta = 0$ we obtain the previous

Hopefully $\quad \begin{matrix} u = \bar{u} + O(\varepsilon) \\ v = \bar{v} + O(\varepsilon) \end{matrix} \quad$ for $\quad 0 \le t \le \frac{T}{\varepsilon}$

Hyperbolic fixed points of * correspond to periodic orbits
" periodic orbits of * " " invariant Tori

Phase Portraits & structural stability

Hale: Duffing's Eq. with small damping and small Harmonic Forcing

Hale Notation

V.5 $$\ddot{u} + \varepsilon\delta\dot{u} + u + \varepsilon\gamma u^3 = \varepsilon B\cos\omega t$$

$$\dot{u} = v$$
$$\dot{v} = -u - \varepsilon\gamma u^3 - \varepsilon\delta v + \varepsilon B\cos\omega t$$

$0 \le \varepsilon$, γ, $\delta \geq 0$, $\omega \geq 0$ real parameters

For $\omega^2 = 1 + \varepsilon\beta$ find conditions which insure $\exists$ periodic solution of period $2\pi/\omega$

Van der Pol Transformation (since we are looking for solutions of period $2\pi/\omega$)

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sin\omega t & \cos\omega t \\ \omega\cos\omega t & -\omega\sin\omega t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \frac{\varepsilon}{\omega}\left(\beta u - \gamma u^3 - \delta v + B\cos\omega t\right)\begin{pmatrix} \cos\omega t \\ -\sin\omega t \end{pmatrix}$$

where $\beta \overset{def}{=} \dfrac{\omega^2 - 1}{\varepsilon}$

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First order averaging for periodic systems

$$\frac{dx}{dt} = \varepsilon f(x,t,\varepsilon) = \varepsilon f_1(x,t) + \varepsilon^2 r(x,t,\varepsilon) \qquad x(0,\varepsilon) = \xi \qquad (1)$$

A) $f : U \times \mathbb{R}_+ \times [0, \varepsilon_0) \longrightarrow \mathbb{C}^n$ $\qquad U \subset \mathbb{C}^n$ open

f is jointly continuous in (x,t) and locally x-Lipschitz uniformly for $\varepsilon \in [0, \varepsilon_0)$

B) $f_1 : U \times \mathbb{R}_+ \to \mathbb{C}^n$

f_1 jointly continuous in (x,t), locally x-Lip on $U \times \mathbb{R}_+$ and 1-periodic in t.

c) $r : U \times \mathbb{R}_+ \times [0, \varepsilon_0) \longrightarrow \mathbb{C}^n$

r satisfies same conditions as f

Averaged Problem

$$\bar{f}_1(x) = \int_0^1 f_1(x,t)\,dt \qquad f_1 \text{ loc } x\text{-Lip} \Rightarrow \bar{f}_1 \text{ loc } x\text{-Lip}$$

$$\frac{dv}{dt} = \varepsilon \bar{f}_1(v) \qquad v(0,\varepsilon) = \xi \qquad (2)$$

Guiding Solution (GS)

$$\frac{du}{d\tau} = \bar{f}_1(u) \qquad u(0) = \xi \qquad (3)$$

The standard existence, uniqueness continuation thm applies to give a maximal forward solution of (3) for $\tau \in [0, T_1(\xi))$ $\ni$ either $T_1(\xi) = \infty$ or "$u \to \partial U$" as $\tau \uparrow T_1(\xi)$.

Pick $T < T_1(\xi)$ in an optimal way for the problem at hand (Basically, any property of the GS on $[0, T]$ will be reflected in (1) on $[0, T/\varepsilon]$).

Thm: (Let $\xi \in U$, then) there exist positive constants ε^* and C $\ni$ for all $0 \le t \le T/\varepsilon$, $0 < \varepsilon \le \varepsilon^*$ the IVP (1) has a unique solution $x(t,\varepsilon)$ in U and

$$\| x(t,\varepsilon) - v(t,\varepsilon) \| \le C\varepsilon . \tag{4}$$

Heuristic Argument

From 1) and 2)

$$x(t,\varepsilon) - v(t,\varepsilon) = \varepsilon \int_0^t \left(f_1(x(s,\varepsilon),s) - f_1(v(s,\varepsilon),s)\right) ds$$

$$+ \varepsilon \int_0^t g(v(s,\varepsilon),s)\, ds + \varepsilon^2 \int_0^t r(x(s,\varepsilon),s,\varepsilon)\, ds \tag{5}$$

$$g(v,s) := f_1(v,s) - \bar{f}_1(v) \qquad \text{Note} \quad \int_0^1 g(v,s)\, ds = 0 \tag{6}$$

$$\therefore \quad \| x(t,\varepsilon) - v(t,\varepsilon) \| \le \varepsilon \int_0^t \| f_1(x(s,\varepsilon),s) - f_1(v(s,\varepsilon),s) \| ds$$

$$+ \varepsilon \left\| \int_0^t g(v(s,\varepsilon),s)\, ds \right\| + \varepsilon^2 \int_0^t \| r(x(s,\varepsilon),s,\varepsilon) \| ds \tag{7}$$

Assume $\| f_1(x,s) - f_1(v,s) \| \le L\|x - v\|$ and $\| r(x,s,\varepsilon) \| \le M_1$.

Under suitable conditions $\| \int_0^t g(v(s,\varepsilon),s)\, ds \| \le M_2(\varepsilon t + 1)$ since $g(v,t)$ has zero t-mean and $v(t,\varepsilon)$ is slowly varying.

$$\therefore \quad \|x(t,\varepsilon) - v(t,\varepsilon)\| \le \overbrace{M_2(\varepsilon^2 t + \varepsilon) + M_1 \varepsilon^2 t}^{= \alpha\varepsilon^2 t + \beta\varepsilon} + \varepsilon L \int_0^t \|x(s,\varepsilon) - v(s,\varepsilon)\|\, ds$$

$$\Rightarrow \quad \|x(t,\varepsilon) - v(t,\varepsilon)\| \le (\alpha\varepsilon^2 t + \beta\varepsilon)\, e^{\varepsilon L t} \le \varepsilon(\alpha T + \beta)\, e^{LT}$$

which gives the result. To make this rigorous we need to avoid the use of global boundedness and global Lipschitz.

<u>Proof of Thm</u>

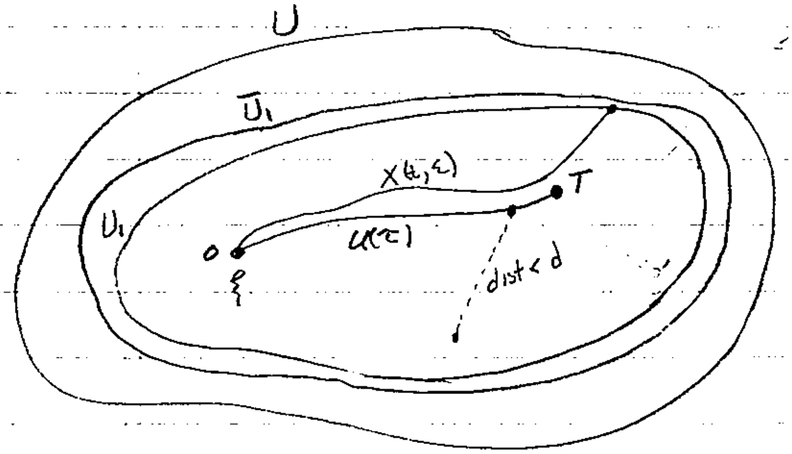
$$S := \{ x \mid x = u(\tau),\ 0 \le \tau \le T \} \subset U$$

Since S is compact and U is open there is a positive distance between S and ∂U, hence we can choose a positive

$$d < \inf \{ \|x - y\| \mid x \in S \text{ and } y \in \mathbb{C}^n - U \}.$$

Let $U_1 = \{ x \in \mathbb{C}^n \mid x = y + z,\ y \in S,\ |z| < d \}$

$\therefore \quad S \subset U_1 \subset \overline{U}_1 \subset U, \quad \overline{U}_1$ compact



$\xi \to T < T_1(\xi) \longrightarrow S$
$\to d \to U_1$

For each ε, f satisfies the condition of e-u-c in U_1. Let $[0, \beta(\varepsilon))$ denote the maximal forward interval of existence of x in U_1. Either $\beta(\varepsilon) = \infty$ or "$x \to \partial U_1$" as $t \nearrow \beta(\varepsilon)$.

Remark: In a specific problem it might be of interest to choose U_1 (i.e. d) and T in an optimal way

Since $\bar{U}_1$ is compact, $\exists$ L_1, L_s, M_r and M_s $\ni$

a) $\|f_1(x,t) - f_1(v,t)\| \le L_1 \|x - v\| \qquad x \in \bar{U}_1,\ v \in S$

b) $\|f_1(v_1,t) - f_1(v_2,t)\| \le L_s \|v_1 - v_2\| \qquad v_1, v_2 \in S$

c) $\|f_1(v,t)\| \le M_s \qquad v \in S$

d) $\|r(x,t,\varepsilon)\| \le M_r \qquad x \in \bar{U}_1$

Now define : $J(\varepsilon) := [0, T/\varepsilon] \cap [0, \beta(\varepsilon))$.

clearly (5) and (7) are valid $\forall\ t \in J(\varepsilon)$, therefore

$$\|x(t,\varepsilon) - v(t,\varepsilon)\| \le \varepsilon^2 M_r t + \varepsilon \left\|\int_0^t g(v(s,\varepsilon), s)\,ds\right\| + \varepsilon L_1 \int_0^t \|x(s,\varepsilon) - v(s,\varepsilon)\|\,ds \qquad (8)$$

the following Lemma completes the proof

Lemma: $\left\|\int_0^t g(v(s),s)\,ds\right\| \le M_s(L_s \varepsilon t + 2)$ (Besjes)

since then

$$\|x(t,\varepsilon) - v(t,\varepsilon)\| \le \varepsilon\left[\varepsilon M_r t + M_s(L_s \varepsilon t + 2)\right] e^{\varepsilon L_1 t}$$

$$\le \varepsilon\left[M_r T + M_s(L_s T + 2)\right] e^{L_1 T}, \quad t \in J(\varepsilon).$$

choose $d^* < d$ and define ε^* via

$$\varepsilon^* [M_r T + M_s (L_s T + 2)] e^{L_1 T} = d^* \quad \text{and} \quad c = \frac{d^*}{\varepsilon^*}$$

thus

$$\|X(t,\varepsilon) - V(t,\varepsilon)\| \le \varepsilon\, C \quad \text{for} \quad 0 < \varepsilon \le \varepsilon^* \text{ and } t \in J(\varepsilon)$$

Assume $\beta(\varepsilon) \le T/\varepsilon$, then $X \not\to \partial U_1$ as $t \nearrow \beta(\varepsilon)$ which contradicts the fact that $[0, \beta(\varepsilon))$ is the maximal forward interval of existence. thus $\beta(\varepsilon) > T/\varepsilon$ and $J(\varepsilon) = [0, T/\varepsilon]$. QED

Proof of Besjes Lemma:

$$\left\| \int_0^t g(v(s),s)\,ds \right\| \le \sum_{b=0}^{n-1} \left\| \int_b^{b+1} [g(v(s),s) - g(v(b),s)]\,ds \right\|$$

$$\int_n^t \| g(v(s),s) \|\,ds \quad \text{since} \quad \int_b^{b+1} g(v(b),s)\,ds = 0$$

$$n = [t]$$

$$\| g(v,s) \| \le \| f_1(v,s) \| + \| \bar{f}_1(v) \| \le 2M_s$$

$$\| g(v_1,s) - g(v_2,s) \| \le \| f_1(v_1,s) - f_1(v_2,s) \| + \| \bar{f}_1(v_1) - \bar{f}_1(v_2) \|$$
$$\le 2L_s$$

$$\therefore \left\| \int_0^t g(v(s),s)\,ds \right\| \le 2L_s \sum_{b=0}^{n-1} \int_b^{b+1} \| v(s) - v(b) \|\,ds + 2M_s$$

Now $\frac{dv}{dt} = \varepsilon \bar{f}_1(v) \Rightarrow \| v(s) - v(b) \| = \| \varepsilon \int_b^s \bar{f}_1(v(\tau))\,d\tau \|$

$$\le \varepsilon M_s |s-b|$$

$$\therefore \int_b^{b+1} \| v(s) - v(b) \|\,ds \le \tfrac{1}{2} \varepsilon M_s$$

$$\therefore \left\| \int_0^t g(v(s),s)\,ds \right\| \le \varepsilon L_s M_s n + 2M_s \le L_s M_s \varepsilon t + 2M_s$$

QED

Compare

B. J. Besjes J. Mécanique 8 (1969), 357–372.

R. Z. Khas'minskii Theory Prob. Appl. 11 (1966), 211–228.