3.5.1 Show that the nonlinear system

$$\left.\begin{aligned}\dot{x} &= -y + xz^2\\ \dot{y} &= x + yz^2\\ \dot{z} &= -z(x^2+y^2)\end{aligned}\right\}$$

has a periodic orbit of $\gamma(t) = (\cos t, \sin t, 0)^T$. Find linearization of this system about $\gamma(t)$, the fundamental matrix $\Phi(t)$ for this system which $\Phi(0) = I$, and Floquet exponents and multipliers of $\gamma(t)$. What are the dimensions of the stable, unstable & center mfds. of $\gamma(t)$?

(Following example 1, p. 221)

Periodic orbit: indeed $\left(-\sin t = -\cancel{\sin t} + 0,\ \cos t = \cos t + 0;\ 0 = 0\right)$ ✓

$$Df = \begin{bmatrix} z^2 & -1 & 2xz \\ z^2 & 1 & 2yz \\ -2xz & -2yz & -(x^2+y^2) \end{bmatrix}; \quad Df(\gamma(t)) = A(t) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad A = VDV^{-1}$$

$$V = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Linearization: $\Phi(t) = e^{At}$

$$A\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \quad V^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$AV = VD \qquad D = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

$$\Rightarrow \Phi(t) = Ve^{Dt}V^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\left[I + D\left(t + \frac{t^3}{3!} + \dots\right) + D^2\left(\frac{t^2}{2} + \frac{t^4}{4!} + \dots\right)\right]\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \Phi(t) = V\begin{pmatrix} 1 & & \\ & e^{t} & \\ & & e^{-t} \end{pmatrix}V^{-1}; \quad \Phi(0) = I$$

~~Characteristic exponents are 0, 1, −1; multipliers 1, $e^{2\pi}$, $e^{-2\pi}$~~

~~The stab~~ However, the zero eigenvalue here and in all similar investigations applies to the direction along the orbit (its eigenvector points along the orbit). Therefore we strike out the corresponding row/column of D with zero evalue and only consider two transverse directions: $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ exponents, $\begin{pmatrix} e^{2\pi} & \\ & e^{-2\pi} \end{pmatrix}$ multipliers

	dimensions
stable	1 + 1 (1 transverse + orbit)
unstable	1
center	0

2. Consider the nonlinear system

$$\left.\begin{aligned} \dot{x} &= x - 4y - \frac{x^3}{4} - xy^2 \\ \dot{y} &= x + y - \frac{x^2 y}{4} - y^3 \\ \dot{z} &= z \end{aligned}\right\}$$

Show that $\gamma(t) = (2\cos 2t, \sin 2t, 0)^T$ is a periodic solution for this system of period $t = \pi$; (show it is a stable limit cycle)*. Determine linearization & show $\dot{x} = Ax \to \Phi(t) = \begin{pmatrix} e^{-2t}\cos 2t & -2\sin 2t & 0 \\ \frac{1}{2}e^{-2t}\sin 2t & \cos 2t & 0 \\ 0 & 0 & e^t \end{pmatrix}$, $\Phi(0) = I$.

Write $\Phi(t) = Q(t)e^{Bt}$, determine exponents, multipliers, and dimensions of invariant manifolds.

let $x = 2r\cos\varphi,\ y = r\sin\varphi,\ z = z$:

$$\begin{matrix} -\sin & \cos \\ + & \\ \cos & \sin \end{matrix} \left(\begin{aligned} 2\dot{r}\cos\varphi - 2r\dot{\varphi}\sin\varphi &= 2r\cos\varphi - 4r\sin\varphi - 8\frac{r^3}{4}\cos^3\varphi - 2r^3\cos\varphi\sin^2\varphi \\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi &= 2r\cos\varphi + r\sin\varphi - 4\frac{r^3}{4}\cos^2\varphi\sin\varphi - r^3\sin^3\varphi \end{aligned} \right\}$$

$$\Rightarrow \dot{r} = r - 3r\cos\varphi\sin\varphi - r^3\left[\frac{1}{4}\cos^2\varphi - \sin^2\varphi\right]$$

$$\Rightarrow \cancel{r}\dot{\varphi} = \cancel{r} + 3\cancel{r}\sin\varphi \Rightarrow \dot{\varphi} = 1 + 3\sin^2\varphi = \frac{5}{2} - \frac{3}{2}\cos 2\varphi > 0$$

$$\begin{matrix} -\sin & \cos \\ \Rightarrow & \\ \cos & \sin \end{matrix} \left(\begin{aligned} \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi &= r\cos\varphi - 2r\sin\varphi - r^3\cos^3\varphi - r^3\cos\varphi\sin^2\varphi \\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi &= 2r\cos\varphi + r\sin\varphi - r^3\cos^2\varphi\sin\varphi - r^3\sin^3\varphi \end{aligned} \right)$$

$$\left.\begin{aligned} \dot{r} &= r - r^3 \\ \cancel{r}\dot{\varphi} &= 2\cancel{r} \Rightarrow \dot{\varphi} = 2 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \dot{r} &= r - r^3 \\ \dot{\varphi} &= 2 \\ \dot{z} &= z \end{aligned}\right.$$

$\dot{z} = z$

$$\nabla \cdot f = \left(1 - \frac{3x^2}{4} - y^2\right) + \left(1 - \frac{x^2}{4} - 3y^2\right)$$
$$= 2 - x^2 - 4y^2\big|_\gamma$$
$$= 2 - 4\cos^2 2t - 4\sin^2 2t = -2$$
$$\Rightarrow \int_0^\pi \nabla \cdot f(\gamma)\,dt = -2\pi < 0 : \text{stable}$$

* Theorem: γ stable/unstable according to $\int_0^T \nabla \cdot f(\gamma(t))\,dt \lessgtr 0$ for 2-dimensional limit cycle γ

$$Df = \begin{bmatrix} 1-\frac{3x^2}{4}-y^2 & -4-2xy & 0 \\ 1-\frac{2xy}{4} & 1-\frac{x^2}{4}-3y^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A(t) = Df(\gamma) = \begin{bmatrix} 1-3\cos^2 2t-\sin^2 2t & -4-4\cos 2t\sin 2t & 0 \\ 1-\cos 2t\sin 2t & 1-\cos^2 2t-3\sin^2 2t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1-\cos 4t & -4-2\sin 4t & 0 \\ 1-\frac{1}{2}\sin 4t & -1+\cos 4t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A\phi = \begin{pmatrix} -1-\cos 4 & -4-2\sin 4 & 0 \\ 1-\frac{1}{2}\sin 4 & -1+\cos 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t}\cos 2t & -2\sin 2t & 0 \\ \frac{1}{2}e^{-2t}\sin 2t & \cos 2t & 0 \\ 0 & 0 & e^t \end{pmatrix}$$

$$= \begin{matrix} e^{-2t}\left((-1-\cos 4)\cos 2 - (2+\sin 4)\sin 2\right) & 2(1+\cos 4)\sin 2 - 2(2+\sin 4)\cos 2 \\ e^{-2t}\left((1-\frac{1}{2}\sin 4)\cos 2 + \frac{1}{2}(-1+\cos 4)\sin 2\right. & -2(1-\frac{1}{2}\sin 4)\sin 2 + (-1+\cos 4)\cos 2 \end{matrix}$$

(1,1) $e^{-2t}\left[-\cos 2 - 2\sin 2 - \overbrace{(\cos 4\cos 2 - \sin 4\sin 2)}^{-\cos 2}\right] = -2e^{-2t}(\cos 2+\sin 2)$

(2,1) $e^{-2t}\left[\cos 2 - \frac{1}{2}\sin 2 - \overbrace{\frac{1}{2}\sin 4\cos 2 + \frac{1}{2}\cos 4\sin 2}^{-\frac{1}{2}\sin 2}\right] = e^{-2t}(\cos 2 - \sin 2)$

(1,2) $2\sin 2 - 4\cos 2 + \underbrace{2\cos 4\sin 2 - 2\sin 4\cos 2}_{\sin(2-4)} = -4\cos 2$

(2,2) $-2\sin 2 - \cos 2 + \overbrace{\sin 4\sin 2 + \cos 4\cos 2}^{\cos 2} = -2\sin 2$

$$A\phi = \begin{pmatrix} -2e^{-2t}(\cos 2+\sin 2) & -4\cos 2 \\ e^{-2t}(\cos 2-\sin 2) & -2\sin 2 \end{pmatrix} = \dot{\Phi}$$

$$\dot{\phi} = \begin{pmatrix} -2e^{-2t}\cos 2t - 2e^{-2t}\sin 2t & -4\cos 2t & \\ -e^{-2t}\sin 2t + e^{-2t}\cos 2t & -2\sin 2t & \\ & & e^t \end{pmatrix} = A\phi$$

Now $\phi = \begin{pmatrix} e^{-2t}\cos 2t & -2\sin 2t & 0 \\ \frac{1}{2}e^{-2t}\sin 2t & \cos 2t & 0 \\ 0 & 0 & e^t \end{pmatrix} = \begin{pmatrix} \cos 2t & -\sin 2t & 0 \\ \sin 2t & \cos 2t & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & ? & \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \cos 2 + 2\sin 2 & 0 \\ -\sin 2 \; 2\cos 2 & 0 \\ 0 \quad 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t}\cos 2 & -2\sin 2 & 0 \\ \frac{1}{2}e^{-2t}\sin 2 & \cos 2 & \\ & & e^t \end{pmatrix} = \begin{pmatrix} e^{-2t} & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{matrix} \cos 2 & 2\sin 2 \\ -\frac{1}{2}\sin 2 & \cos 2 \end{matrix}$$

neglect 0 eigenvalue

$$\Phi = \underbrace{\begin{pmatrix} \cos 2t & -2\sin 2t & 0 \\ \frac{1}{2}\sin 2t & \cos 2t & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\substack{Q(t) \\ \pi\text{-periodic}}} \begin{pmatrix} e^{-2t} & & \\ & & \\ & & e^t \end{pmatrix} = e^{\begin{pmatrix} -2 & \\ & 1 \end{pmatrix} t}$$

exponents $-2, 1$

multipliers $e^{-2\pi}, e^{\pi}$

stable	1 + 1 (count trajectory as stable)
unstable	1
center	0