Texts in Applied Mathematics 7

Lawrence Perko

Differential Equations and Dynamical Systems

Solutions Manual

Springer

Problem Solutions for Differential Equations and Dynamical Systems

Solutions Manual for TAM 7, Differential Equations and Dynamical Systems, 3rd edition

by
Lawrence Perko

The Solutions Manual is free to all lecturers who adopt TAM 7 for their course. It is also free to any students who have a written approval from their lecturer.

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Problem Solutions for Differential Equations and Dynamical Systems

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PREFACE

This set of problem solutions for the 3rd edition of the author's book *Differential Equations and Dynamical Systems* is intended as an aid for students working on the problem sets that appear at the end of each section in the book. Most of the details necessary to obtain the solutions, along with the solutions themselves, are given for most of the problems in the book. Any additions, corrections or innovative methods of solution should be sent directly to the author, Lawrence Perko, Department of Mathematics, Northern Arizona University, Flagstaff, Arizona 86011 or to Lawrence.Perko@NAU.EDU. The author would like to take this opportunity to thank Louella Holter for her patience and precision in typing the camera-ready copy for this solutions manual.

CONTENTS

1. LINEAR SYSTEMS

PROBLEM SET 1.1

Let $\mathbf{x} = (x_1, x_2, x_3)^T = (x, y, z)^T$ and $\mathbf{x}(0) = (x_0, y_0, z_0)^T$.

1. (a) $x(t) = x_0 e^t$, $y(t) = y_0 e^t$, and solution curves lie on the straight lines $y = (y_0/x_0)x$ or on the y-axis. The phase portrait is given in Problem 3 below with $a = 1$.

(b) $x(t) = x_0 e^t$, $y(t) = y_0 e^{2t}$, and solution curves, other than those on the x and y axes, lie on the parabolas $y = (y_0/x_0^2)x^2$. Cf. Problem 3 below with $a = 2$.

(c) $x(t) = x_0 e^t$, $y(t) = y_0 e^{3t}$, and solution curves lie on the curves $y = (y_0/x_0^3)x^3$.

(d) $\dot{x} = -y$, $\dot{y} = x$ can be written as $\ddot{y} = \dot{x} = -y$ or $\ddot{y} + y = 0$ which has the general solution $y(t) = c_1 \cos t + c_2 \sin t$; thus, $x(t) = \dot{y}(t) = -c_1 \sin t + c_2 \cos t$; or in terms of the initial conditions $x(t) = x_0 \cos t - y_0 \sin t$ and $y(t) = x_0 \sin t + y_0 \cos t$. It follows that for all $t \in \mathbf{R}$, $x^2(t) + y^2(t) = x_0^2 + y_0^2$ and solution curves lie on these circles. Cf. Figure 4 in Section 1.5.

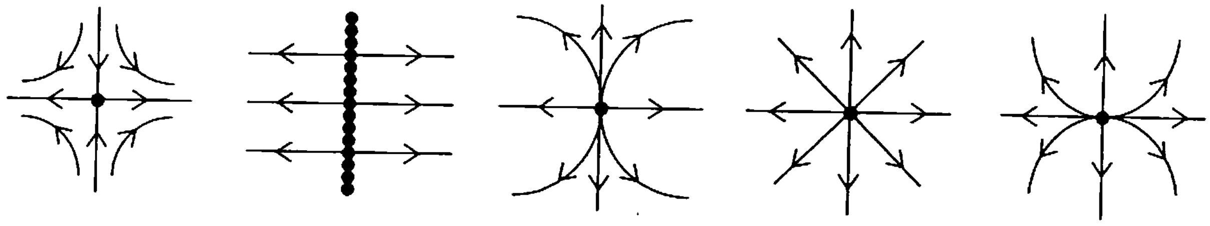
(e) $y(t) = c_2 e^{-t}$ and then solving the first-order linear differential equation $\dot{x} + x = c_2 e^{-t}$ leads to $x(t) = c_1 e^{-t} + c_2 t e^{-t}$ with $c_1 = x_0$ and $c_2 = y_0$. Cf. Figure 2 with $\lambda < 0$ in Section 1.5.

2. (a) $x(t) = x_0 e^t$, $y(t) = y_0 e^t$, $z(t) = z_0 e^t$, and $E^u = \mathbf{R}^3$.

(b) $x(t) = x_0 e^{-t}$, $y(t) = y_0 e^{-t}$, $z(t) = z_0 e^t$, $E^s = \text{Span}\{(1, 0, 0)^T, (0, 1, 0)^T\}$, and $E^u = \text{Span}\{(0, 0, 1)\}$. Cf. Figure 3 with the arrows reversed.

(c) $x(t) = x_0 \cos t - y_0 \sin t$, $y(t) = x_0 \sin t + y_0 \cos t$, $z(t) = z_0 e^{-t}$; solution curves lie on the cylinders $x^2 + y^2 = c^2$ and approach circular periodic orbits in the x,y plane as $t \to \infty$; $E^c = \text{Span}\{(1, 0, 0)^T, (0, 1, 0)^T\}$, $E^s = \text{Span}\{(0, 0, 1)^T\}$.

3. $x(t) = x_0e^t$, $y(t) = y_0e^{at}$.



a = −1 a = 0 a = 1/2 a = 1 a = 2

4. $x_1(t) = x_{10}e^{\lambda_1 t}$, $x_2(t) = x_{20}e^{\lambda_2 t}$, $\cdots$, $x_n(t) = x_{n0}e^{\lambda_n t}$. Thus, $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$ for all $\mathbf{x}_0 \in \mathbf{R}^n$ if $\lambda_1 < 0, \cdots, \lambda_n < 0$ (and also if $R_e(\lambda_j) < 0$ for $j = 1, 2, \cdots, n$).

5. If $k > 0$, the vectors $A\mathbf{x}$ and $kA\mathbf{x}$ point in the same direction and they are related by the scale factor k. If $k < 0$, the vectors $A\mathbf{x}$ and $kA\mathbf{x}$ point in opposite directions and are related by the scale factor $|k|$.

6. (a) $\dot{\mathbf{w}}(t) = a\dot{\mathbf{u}}(t) + b\dot{\mathbf{v}}(t) = aA\mathbf{u}(t) + bA\mathbf{v}(t) = A[a\mathbf{u}(t) + b\mathbf{v}(t)] = A\mathbf{w}(t)$ for all $t \in \mathbf{R}$.

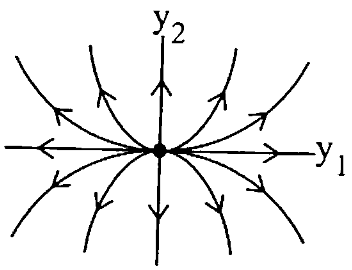
(b) $\mathbf{u}(t) = (e^t, 0)^T$, $\mathbf{v}(t) = (0, e^{-2t})^T$ and the general solution of $\dot{\mathbf{x}} = A\mathbf{x}$ is given by $\mathbf{x}(t) = x_0\mathbf{u}(t) + y_0\mathbf{v}(t)$.

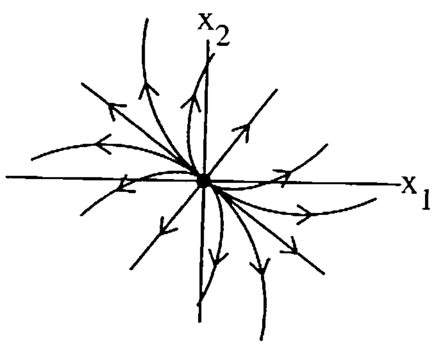
PROBLEM SET 1.2

1. (a) $\lambda_1 = 2$, $\lambda_2 = 4$, $\mathbf{v}_1 = (1, -1)^T$, $\mathbf{v}_2 = (1, 1)^T$, $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $P^{-1} = 1/2\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and

$$B = P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\mathbf{y}(t) = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix}\mathbf{y}_0, \qquad \mathbf{x}(t) = P\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix}P^{-1}\mathbf{x}_0 = 1/2\begin{bmatrix} e^{2t}+e^{4t} & e^{4t}-e^{2t} \\ e^{4t}-e^{2t} & e^{4t}+e^{2t} \end{bmatrix}\mathbf{x}_0.$$



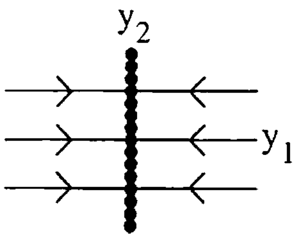


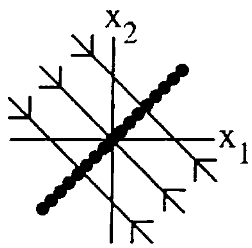
(b) $\lambda_1 = 4, \lambda_2 = -2, \mathbf{v}_1 = (1, 1)^T, \mathbf{v}_2 = (1, -1)^T,$

$$\mathbf{y}(t) = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \mathbf{y}_0, \qquad \mathbf{x}(t) = 1/2 \begin{bmatrix} e^{4t} + e^{-2t} & e^{4t} - e^{-2t} \\ e^{4t} - e^{-2t} & e^{4t} + e^{-2t} \end{bmatrix} \mathbf{x}_0.$$

(c) $\lambda_1 = -2, \lambda_2 = 0, \mathbf{v}_1 = (1, -1)^T, \mathbf{v}_2 = (1,1)^T,$

$$\mathbf{y}(t) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & 1 \end{bmatrix} \mathbf{y}_0, \qquad \mathbf{x}(t) = 1/2 \begin{bmatrix} e^{-2t} + 1 & 1 - e^{-2t} \\ 1 - e^{-2t} & 1 + e^{-2t} \end{bmatrix} \mathbf{x}_0.$$





2. $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1, \mathbf{v}_1 = (2, -2, 1)^T, \mathbf{v}_2 = (0, 1, 0)^T, \mathbf{v}_3 = (0, 0, 1)^T$

$$\mathbf{y}(t) = \begin{bmatrix} e^{t} & & \\ & e^{2t} & \\ & & e^{-t} \end{bmatrix} \mathbf{y}_0, \quad \mathbf{x}(t) = 1/2 \begin{bmatrix} 2e^{t} & 0 & 0 \\ 2(e^{2t} - e^{t}) & 2e^{2t} & 0 \\ e^{t} - e^{-t} & 0 & 2e^{-t} \end{bmatrix} \mathbf{x}_0,$$

$E^s = \text{Span}\ \{\mathbf{v}_3\},\ E^u = \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2\}.$

3. $\dot{\mathbf{x}} = A\mathbf{x}$

(a) $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (c) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}.$

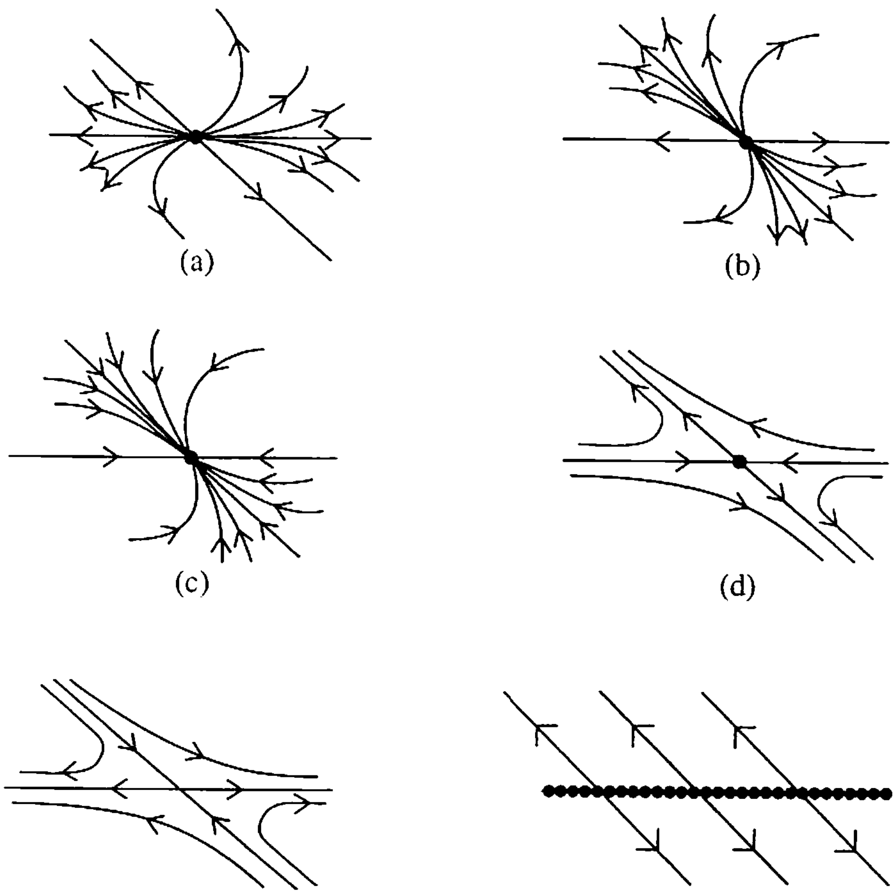
4. (a) $\mathbf{x}(t) = 1/2(3e^{4t} - e^{2t}, 3e^{4t} + e^{2t})$ (b) $\mathbf{x}(t) = 1/2(2e^{t}, 6e^{2t} - 2e^{t}, e^{t} + 5e^{-t}).$

5. $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ iff $\lambda_j < 0$ for $j = 1, 2, 3, \cdots, n.$

6. $$\phi(t, x_0) = P\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix}P^{-1}x_0 \text{ and } \lim_{y_0\to x_0}\phi(t, y_0) = \phi(t, x_0) \text{ since } \lim_{y_0\to x_0} y_0 = x_0$$

according to the definition of the limit.

7.



(a) (b) (c) (d) (e) (f)

PROBLEM SET 1.3

1. (a) $\|A\| = \max_{|x|\le 1}|Ax| = \max_{|x|\le 1}\sqrt{4x^2+9y^2} \le 3|x|$; but for $x = (0,1)^T$, $|Ax| = |-3| = 3$; thus, $\|A\| = 3$.

(b) Following the hint for (c), we can maximize $|Ax|^2 = x^2 + 4xy + 5y^2$ subject to the constraint $x^2 + y^2 = 1$ to find $x^2 = (2 \pm \sqrt{2})/4$ and $y^2 = 1 - x^2$ which leads to $\|A\| = 2.4142136\cdots$; or since $AA^T = \begin{bmatrix} 1 & 2 \\ 2 & 5\end{bmatrix}$ with eigenvalues $3 \pm 2\sqrt{2}$, we have $\|A\| = \sqrt{3+2\sqrt{2}} = 1 + \sqrt{2}$.

(c) We can either maximize $|\mathbf{Ax}|^2 = 26x^2 + 10xy + y^2$ subject to the constraint $x^2 + y^2 = 1$; or find the eigenvalues of $AA^T = \begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$ which are $(27 \pm \sqrt{725})/2$; in either case, $\|A\| = 5.1925824\cdots$.

2. By definition, $\|T\| = \max\limits_{|\mathbf{x}| \le 1} |T(\mathbf{x})|$. Thus, $\|T\| \ge \max\limits_{|\mathbf{x}| = 1} |T(\mathbf{x})|$. But $\max\limits_{|\mathbf{x}| = 1} |T(\mathbf{x})| = \sup\limits_{\mathbf{x} \ne 0} \dfrac{|T(\mathbf{x})|}{|\mathbf{x}|}$ since if $|\mathbf{x}| = a$ and we set $\mathbf{y} = \mathbf{x}/a$ for $\mathbf{x} \ne \mathbf{0}$, then $|\mathbf{y}| = |\mathbf{x}|/a = 1$ and since T is linear, $\sup\limits_{\mathbf{x} \ne 0} \dfrac{|T(\mathbf{x})|}{|\mathbf{x}|} = \sup\limits_{\mathbf{x} \ne 0} \dfrac{|T(\mathbf{x})|}{a} = \sup\limits_{\mathbf{x} \ne 0} \left|T(\dfrac{\mathbf{x}}{a})\right| = \max\limits_{|\mathbf{y}| = 1} |T(\mathbf{y})|$. Thus, $\|T\| \le \sup\limits_{0<|\mathbf{x}|\le 1} \dfrac{|T(\mathbf{x})|}{|\mathbf{x}|} \le \sup\limits_{\mathbf{x} \ne 0} \dfrac{|T(\mathbf{x})|}{|\mathbf{x}|} = \max\limits_{|\mathbf{x}| = 1} |T(\mathbf{x})|$. It follows that $\|T\| = \max\limits_{|\mathbf{x}| = 1} |T(\mathbf{x})| = \sup\limits_{\mathbf{x} \ne 0} |T(\mathbf{x})|/|\mathbf{x}|$.

3. If T is invertible, then there exists an inverse, T^{-1}, such that $TT^{-1} = I$ and therefore $\|TT^{-1}\| = 1$. By the lemma in Section 3, $1 = \|TT^{-1}\| \le \|T\|\ \|T^{-1}\|$. This implies that $\|T\| > 0$, $\|T^{-1}\| > 0$, and $\|T^{-1}\| \ge \dfrac{1}{\|T\|}$.

4. Given $T \in L(\mathbf{R}^n)$ with $\|I - T\| < 1$. Let $a = \|I - T\| < 1$ and the geometric series Σa^k converges. Thus, by the Weierstrass M-Test, $\sum\limits_{k=0}^{\infty}(I - T)^k$ converges absolutely to $S \in L(\mathbf{R}^n)$. By induction it follows that $T\left[I + (I - T) + \cdots + (I - T)^n\right] = I - (I - T)^{n+1}$. Thus, $TS = T\sum\limits_{k=0}^{\infty}(I - T)^k = \sum\limits_{k=0}^{\infty}T(I - T)^k = \lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}T(I - T)^k = \lim\limits_{n\to\infty}\left[I - (I - T)^{n+1}\right] = I$ since $\lim\limits_{n\to\infty}\|I - T\|^{n+1} = 0$ which implies that $\lim\limits_{n\to\infty}(I - T)^{n+1} = 0$ since $0 \le \|(I - T)^{n+1}\| \le \|(I - T)\|^{n+1}$. Therefore $S = T^{-1}$.

5. (a) $e^A = \begin{bmatrix} e^2 & 0 \\ 0 & e^{-3} \end{bmatrix}$.

(b) The eigenvalues and eigenvectors of A are $\lambda_1 = 1$, $\lambda_2 = -1$, $\mathbf{v}_1 = (1, 0)^T$, $\mathbf{v}_2 = (-1, 1)^T$; thus, $e^A = P\begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix}P^{-1} = \begin{bmatrix} e & e - e^{-1} \\ 0 & e^{-1} \end{bmatrix}$ where $P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

(c) $e^A = e\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ by Corollary 4.

(d) The eigenvalues and eigenvectors of A are $\lambda_1 = 2$, $\lambda_2 = -1$, $\mathbf{v}_1 = (2, 1)^T$, $\mathbf{v}_2 = (1, 1)^T$; thus, $e^A = P\begin{bmatrix} e^2 & 0 \\ 0 & e^{-1} \end{bmatrix}P^{-1} = \begin{bmatrix} 2e^2 - e^{-1} & 2e^{-1} - 2e^2 \\ e^2 - e^{-1} & 2e^{-1} - e^2 \end{bmatrix}$ where $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

(e) $e^A = e^2\begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix}$ by Corollary 3.

(f) The eigenvalues and eigenvectors of A are $\lambda_1 = 1$, $\lambda_2 = -1$, $\mathbf{v}_1 = (1, 1)^T$, $\mathbf{v}_2 = (-1, 1)^T$; thus $e^A = P\begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix}P^{-1} = \begin{bmatrix} \cosh(1) & \sinh(1) \\ \sinh(1) & \cosh(1) \end{bmatrix}$ with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Note that $A^2 = I$ and from Definition 2 it therefore follows that $e^A = I(1 + 1/2! + 1/4! + \cdots) + A(1 + 1/3! + 1/5! + \cdots) = I\cosh(1) + A\sinh(1)$. This remark also applies to part (b).

6. (a) The eigenvalues are e^2, e^{-3}; e, e^{-1}; e, e; e^2, e^{-1}; $e^{2\pm i} = e^2[\cos(1) \pm i\sin(1)]$; e, e^{-1}.

(b) If $A\mathbf{x} = \lambda\mathbf{x}$, then $e^A\mathbf{x} = \lim_{k\to\infty}[I + A + A^2/2! + \cdots + A^k/k!]\mathbf{x} = \lim_{k\to\infty}[\mathbf{x} + \lambda\mathbf{x} + \lambda^2\mathbf{x}/2! + \cdots + \lambda^k\mathbf{x}/k!] = e^\lambda\mathbf{x}$.

(c) If $A = P\,\mathrm{diag}\,[\lambda_j]\,P^{-1}$, then by Corollary 1, $\det e^A = \det\{P\,\mathrm{diag}\,[e^{\lambda_j}]\,P^{-1}\} = \det\{\mathrm{diag}\,[e^{\lambda_j}]\} = e^{\lambda_1}\cdots e^{\lambda_k} = e^{\mathrm{trace}A}$. For a 2 x 2 matrix A with repeated eigenvalues λ, we have $\det e^A = \det\begin{bmatrix} e^\lambda & e^\lambda \\ 0 & e^\lambda \end{bmatrix} = e^{2\lambda} = e^{\mathrm{trace}A}$; and for a 2 x 2 matrix A with complex eigenvalues, $\lambda = a \pm ib$, we have $\det e^A = \det\begin{bmatrix} e^a\cos b & -e^a\sin b \\ e^a\sin b & e^a\cos b \end{bmatrix} = e^{2a} = e^{\mathrm{trace}A}$ (since the trace $A = \lambda_1 + \lambda_2 = (a + ib) + (a - ib) = 2a$ in this case).

7. (a) $e^A = \mathrm{diag}[e, e^2, e^3]$.

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = N + S$ and $NS = SN$ so that by Proposition 2,

$$e^A = \mathrm{diag}\,[e, e^2, e^2]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e & 0 & 0 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{bmatrix}$$ since $N^2 = 0$ implies that $e^N = I + N$.

(c) $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = N + S$ and $NS = SN$ so that by Proposition 2

$e^A = e^S e^N = e^2 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix}$ since $N^3 = 0$ implies that $e^N = I + N + N^2/2$.

8. For $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ we have $AB = 0 \neq BA = B$, $e^{A+B} = \begin{bmatrix} e & 0 \\ e-1 & 1 \end{bmatrix} \neq e^A e^B = \begin{bmatrix} e & 0 \\ 1 & 1 \end{bmatrix}$.

9. If $T(\mathbf{x}) \in E$ for all $\mathbf{x} \in E$, then by induction $T^2(\mathbf{x}) \in E, \cdots, T^k(\mathbf{x}) \in E$ and therefore $e^T(\mathbf{x}) = \lim_{k\to\infty} [I + T + \cdots + T^k/k!]\mathbf{x} = \lim_{k\to\infty} [\mathbf{x} + T(\mathbf{x}) + \cdots + \frac{T^k(\mathbf{x})}{k!}] \in E$ since any subspace E of $\mathbf{R}^n$ is complete and since $\mathbf{x}_k = \mathbf{x} + T(\mathbf{x}) + \cdots + T^k(\mathbf{x})/k!$ is a Cauchy sequence in E.

Problem Set 1.4

1. (a) $\mathbf{x}(t) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\mu t} \end{bmatrix}$

(b) $\mathbf{x}(t) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} x_0 e^{\lambda t} + y_0 t e^{\lambda t} \\ y_0 e^{\lambda t} \end{bmatrix}$

(c) $\mathbf{x}(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \mathbf{x}_0 = e^{at} \begin{bmatrix} x_0 \cos bt - y_0 \sin bt \\ x_0 \sin bt + y_0 \cos bt \end{bmatrix}$

2. $\mathbf{x}(t) = e^{-t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \mathbf{x}_0$

3. (a) $\lambda_1 = 2, \lambda_2 = 4, \mathbf{v}_1 = (1, -1)^T, \mathbf{v}_2 = (1, 1)^T, \mathbf{x}(t) = P \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix} P^{-1}\mathbf{x}_0 =$

$$1/2\begin{bmatrix} e^{2t}+e^{4t} & e^{4t}-e^{2t} \\ e^{4t}-e^{2t} & e^{4t}+e^{2t} \end{bmatrix}\mathbf{x}_0 = e^{3t}\begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}\mathbf{x}_0 \text{ where } P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

(b) $\lambda_1 = 4, \lambda_2 = -2, \mathbf{v}_1 = (1, 1)^T, \mathbf{v}_2 = (1, -1)^T, \mathbf{x}(t) = P\begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} P^{-1} =$

$$1/2\begin{bmatrix} e^{4t}+e^{-2t} & e^{4t}-e^{-2t} \\ e^{4t}-e^{-2t} & e^{4t}+e^{2t} \end{bmatrix}\mathbf{x}_0 = e^{t}\begin{bmatrix} \cosh 3t & \sinh 3t \\ \sinh 3t & \cosh 3t \end{bmatrix}\mathbf{x}_0 \text{ where } P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

4. From Problem 2 in Problem Set 2, $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = P\,\mathrm{diag}\,[e^{\lambda_i t}]\,P^{-1} =$

$$1/2\begin{bmatrix} 2e^{t} & 0 & 0 \\ 2e^{2t}-2e^{t} & 2e^{2t} & 0 \\ e^{t}-e^{-t} & 0 & 2e^{-t} \end{bmatrix}\mathbf{x}_0 \text{ where } P = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } P^{-1} = 1/2\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

5. (a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = S + N$ where S and N commute. Thus,

$$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = e^{2t}\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}\mathbf{x}_0.$$

(b) $\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{2t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\mathbf{x}_0.$

(c) $\lambda_1 = 1, \lambda_2 = -1, \mathbf{v}_1 = (1, 1)^T, \mathbf{v}_2 = (-1, 1)^T, \mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = P^{-1}\begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} P\mathbf{x}_0 =$

$$1/2\begin{bmatrix} e^{t}+e^{-t} & e^{t}-e^{-t} \\ e^{t}-e^{-t} & e^{t}+e^{-t} \end{bmatrix}\mathbf{x}_0 = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}\mathbf{x}_0 \text{ where } P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, P^{-1} = 1/2\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

(d) $A = -2I + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = S + N$ where S and N commute. Thus, $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 =$

$$e^{-2t}\,[I + Nt + N^2t^2/2]\mathbf{x}_0 = e^{-2t}\begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2/2 & t & 1 \end{bmatrix}\mathbf{x}_0.$$

6. Since $T(\mathbf{x}) \in E$ for all $\mathbf{x} \in E$ and since $T(\mathbf{x}) = A\mathbf{x}$, it follows that if $\mathbf{x}_0 \in E$ then $A\mathbf{x}_0 \in E$ and $tA\mathbf{x}_0 \in E$ since E is a linear subspace of $\mathbf{R}^n$. It then follows by induction that $(t^k/k!)A^k\mathbf{x}_0 \in E$ for all $k \in \mathbf{N}$. Therefore $\sum_{k=0}^{N} A^k t^k \mathbf{x}_0/k! \in E$ since E is a linear subspace of $\mathbf{R}^n$. Then since a closed subset of a complete metric space is complete, it follows that E is a complete normed linear space; i.e., every Cauchy sequence in E converges to a vector in E. (Cf. Theorem 3.11, p. 53 in [R].) Thus, for all $t \in \mathbf{R}$ $\lim_{N\to\infty} \sum_{k=0}^{N} A^k t^k \mathbf{x}_0/k! = e^{At}\mathbf{x}_0 \in E$. And therefore by the Fundamental Theorem for linear systems $\mathbf{x}(t) = e^{At}\mathbf{x}_0 \in E$ for all $t \in \mathbf{R}$.

7. Suppose that there is a $\lambda < 0$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some $\mathbf{v} \neq \mathbf{0}$. Then $\mathbf{x}(t) = e^{At}\mathbf{v}$ is a solution of (1) with $\mathbf{x}(0) = \mathbf{v}$. But $e^{At}\mathbf{v} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}\mathbf{v} = \sum_{k=0}^{\infty} \frac{t^k \lambda^k}{k!}\mathbf{v} = e^{\lambda t}\mathbf{v}$ since, by induction, $A^k\mathbf{v} = \lambda^k\mathbf{v}$. Thus, $\lim_{t\to\infty} \mathbf{x}(t) = \lim_{t\to\infty} e^{At}\mathbf{v} = \lim_{t\to\infty} e^{\lambda t}\mathbf{v} = \mathbf{0}$ since $\lambda < 0$.

8. By the Fundamental Theorem for linear systems, the solution of $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is given by $\phi(t, \mathbf{x}_0) = e^{At}\mathbf{x}_0$. Thus, for all $t \in \mathbf{R}$, $\lim_{\mathbf{y}\to\mathbf{x}_0} \phi(t,\mathbf{y}) = \lim_{\mathbf{y}\to\mathbf{x}_0} e^{At}\mathbf{y} = e^{At} \lim_{\mathbf{y}\to\mathbf{x}_0} \mathbf{y} = e^{At}\mathbf{x}_0 = \phi(t,\mathbf{x}_0)$.

PROBLEM SET 1.5

1. (a) $\delta = -2 < 0$ implies that (1) has a saddle at the origin.

   (b) $\delta = 8$, $\tau = 6$, $\tau^2 - 4\delta = 4 > 0$ implies that (1) has an unstable node at the origin.

   (c) $\delta = 2$, $\tau = 0$ implies that (1) has a center at the origin.

   (d) $\delta = 5$, $\tau = 4$, $\tau^2 - 4\delta = -4$ implies that (1) has an unstable focus at the origin.

   (e) $\delta = \lambda^2 + 2 > 0$, $\tau = 2\lambda$, $\tau^2 - 4\delta = -8 < 0$ implies that for $\lambda \neq 0$ (1) has a focus at the origin; it is stable if $\lambda < 0$ and unstable if $\lambda > 0$; and (1) has a center at the origin if $\lambda = 0$.

   (f) $\delta = \lambda^2 - 2$, $\tau = 2\lambda$, $\tau^2 - 4\delta = 8 > 0$ implies that (1) has a saddle at the origin if $|\lambda| < \sqrt{2}$; (1) has a node at the origin if $|\lambda| > \sqrt{2}$; it is stable if $\lambda < -\sqrt{2}$ and unstable if $\lambda > \sqrt{2}$; and (1) has a degenerate critical point at the origin if $|\lambda| = \sqrt{2}$.

2. (a) $x_1(t) = x_1(0)e^{3t}$, $x_2(t) = x_2(0)e^{3t}$. Cf. Problem 3 with a = 1 in Problem Set 1.1.

(b) $x_1(t) = x_1(0)e^{3t}$, $x_2(t) = x_2(0)e^{t}$. Cf. Problem 3 with a = 1/2 in Problem Set 1.1.

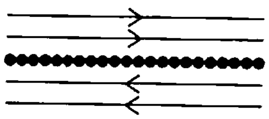
(c) $x_1(t) = x_1(0)e^{t}$, $x_2(t) = x_2(0)e^{3t}$. Cf. Problem 3 with a = 2 in Problem Set 1.1.

(d) $x_1(t) = [x_1(0) + x_2(0)t]e^t$, $x_2(t) = x_2(0)e^t$, which follows from $e^{At}\mathbf{x}_0 = e^t\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\mathbf{x}_0$. Cf. Figure 2.

3. $\delta = 2a + b^2 > 0$ iff $a > -b^2/2$ and $\tau = a + 2 < 0$ iff $a < -2$. Thus, the system $\dot{\mathbf{x}} = A\mathbf{x}$ has a sink at the origin iff $-b^2/2 < a < -2$.

4. (a) $x(t) = x_0e^{\lambda t}$, $y(t) = y_0$. For $\lambda > 0$ cf. Problem 3 with a = 0 in Problem Set 1.1.

(b) $x(t) = x_0 + y_0t$, $y(t) = y_0$.



(c) $x(t) = x_0$, $y(t) = y_0$; every point $\mathbf{x}_0 \in \mathbf{R}^2$ is a critical point.

5. The second-order differential equation can be written in the form of a linear system (1) with $A = \begin{bmatrix}0 & -1\\ b & a\end{bmatrix}$. If $b < 0$, the origin is a saddle; if $b > 0$ and $a^2 - 4b \geq 0$, the origin is a node which is stable if $a < 0$ and unstable if $a > 0$; if $b > 0$, $a^2 - 4b < 0$ and $a \neq 0$, the origin is a focus which is stable if $a < 0$ and unstable if $a > 0$; if $b > 0$ and $a = 0$, the origin is a center; and if $b = 0$, the origin is a degenerate critical point.

6. $x_1(t) = x_1(0)e^t$, $x_2(t) = x_1(0)e^t + [x_2(0) - x_1(0)]e^{2t}$; $\lambda_1 = 1$, $\lambda_2 = 2$, $\mathbf{v}_1 = (1, 1)^T$ and $\mathbf{v}_2 = (0, 1)^T$; the origin is an unstable node.

7. $\lambda_1 = (5 + \sqrt{33})/2$, $\lambda_2 = (5 - \sqrt{33})/2$, $\mathbf{v}_1 = (4, 3 + \sqrt{33})^T$, $\mathbf{v}_2 = (4, 3 - \sqrt{33})^T$; the separatrices are the four trajectories in $E^s \cup E^u$ and the origin.

8. Since $x_1(t) = x_1(0)\cos t - x_2(0)\sin t$ and $x_2(t) = x_1(0)\sin t + x_2(0)\cos t$, $r(t) = \sqrt{x_1^2(t) + x_2^2(t)} = \sqrt{x_1^2(0) + x_2^2(0)}$, a constant and $\theta(t) = \tan^{-1}[x_2(t)/x_1(t)] = \tan^{-1}[x_2(0)/x_1(0)] + t$; the origin is a center for this system.

9. Differentiating $r^2 = x_1^2 + x_2^2$ with respect to t leads to $2r\dot{r} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$ or $\dot{r} = (x_1\dot{x}_1 + x_2\dot{x}_2)/r$ for $r \neq 0$. Differentiating $\theta = \tan^{-1}(x_2/x_1)$ with respect to t leads to $\dot{\theta} = (x_1\dot{x}_2 - x_2\dot{x}_1)/x_1^2[1 + (x_2/x_1)^2] = (x_1\dot{x}_2 - x_2\dot{x}_1)/r^2$ for $r \neq 0$. For the system in Problem 8 we easily obtain $\dot{r} = ar$ and $\dot{\theta} = b$ from these equations. These latter equations with the initial conditions $r(0) = r_0$ and $\theta(0) = \theta_0$ have the solution $r(t) = r_0e^{at}$, $\theta(t) = \theta_0 + bt$. Thus for $a < 0$, $r(t) \to 0$ as $t \to \infty$ and for $b > 0$ (or $b < 0$), $\theta(t) \to \infty$ as $t \to \infty$ (or as $t \to -\infty$) as in Figure 3. And for $a = 0$, $r(t) = r_0$ while $\theta(t) \to \infty$ as $t \to \infty$ (or as $t \to -\infty$) for $b > 0$ (or $b < 0$) as in Figure 4.

PROBLEM SET 1.6

1. $\lambda = 2 \pm i$. For $\lambda = 2 + i$, $\mathbf{w} = \mathbf{u} + i\mathbf{v} = (1, 1)^T + i(1, 0)^T$,

$$P = [\mathbf{v}\ \mathbf{u}] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},\ P^{-1}AP = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

and the solution $\mathbf{x}(t) = Pe^{2t}R_tP^{-1}\mathbf{x}_0 = e^{2t}\begin{bmatrix} \cos t + \sin t & -2\sin t \\ \sin t & \cos t - \sin t \end{bmatrix}\mathbf{x}_0$

where $R_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$.

2. $\lambda = 1 \pm i$, $\lambda_3 = -2$, $\mathbf{w} = (1 - i, -1, 0)^T$, $\mathbf{v}_3 = (0, 0, 1)^T$, $P = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$P^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the solution $\mathbf{x}(t) = P\begin{bmatrix} e^t\cos t & -e^t\sin t & 0 \\ e^t\sin t & e^t\cos t & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}P^{-1}\mathbf{x}_0 =$

$$\begin{bmatrix} e^t(\cos t - \sin t) & -2e^t\sin t & 0 \\ e^t\sin t & e^t(\sin t + \cos t) & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}\mathbf{x}_0.$$

3. $\lambda_1 = 1, \lambda = 2 \pm 3i, \mathbf{v}_1 = (-10, 3, 1), \mathbf{w} = (0, i, 1)^T, P = \begin{bmatrix} -10 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$,

$P^{-1} = \begin{bmatrix} -1/10 & 0 & 0 \\ 3/10 & 1 & 0 \\ 1/10 & 0 & 1 \end{bmatrix}$ and the solution $\mathbf{x}(t) = P \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t}\cos 3t & -e^{2t}\sin 3t \\ 0 & e^{2t}\sin 3t & e^{2t}\cos 3t \end{bmatrix} P^{-1}\mathbf{x}_0 =$

$$\begin{bmatrix} e^t & 0 & 0 \\ (-3e^t + 3e^{2t}\cos 3t - e^{2t}\sin 3t)/10 & e^{2t}\cos 3t & -e^{2t}\sin 3t \\ (-e^t + 3e^{2t}\sin 3t + e^{2t}\cos 3t)/10 & e^{2t}\sin 3t & e^{2t}\cos 3t \end{bmatrix} \mathbf{x}_0.$$

4. $\lambda_1 = -1 + i, \lambda_3 = 1 + i, \mathbf{w}_1 = (1, -i, 0, 0)^T, \mathbf{w}_3 = (0, 0, 1 - i, -1)^T$,

$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, P^{-1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ and the solution $\mathbf{x}(t) =$

$$P \begin{bmatrix} e^{-t}R_t & 0 \\ 0 & e^t R_t \end{bmatrix} P^{-1}\mathbf{x}_0 = \begin{bmatrix} e^{-t}\cos t & -e^{-t}\sin t & 0 & 0 \\ e^{-t}\sin t & e^{-t}\cos t & 0 & 0 \\ 0 & 0 & e^t(\cos t - \sin t) & -2e^t \sin t \\ 0 & 0 & e^t \sin t & e^t(\sin t + \cos t) \end{bmatrix} \mathbf{x}_0.$$

PROBLEM SET 1.7

1. (a) $\lambda_1 = \lambda_2 = 1$; $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = S + N$ where S and N commute and $N^2 = 0$:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{St}e^{Nt}\mathbf{x}_0 = e^t \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} \mathbf{x}_0.$$

(b) $\lambda_1 = \lambda_2 = 2$: $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = S + N$ where S and N commute and $N^2 = 0$:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{St}e^{Nt}\mathbf{x}_0 = e^{2t} \begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix} \mathbf{x}_0.$$

(c) $\lambda_1 = 1, 1$: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = S + N$; $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = e^{St}\,e^{Nt}\,\mathbf{x}_0 = e^t \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}_0$.

(d) $\lambda_1 = 1, \lambda_2 = -1$: $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = S + N$, but S and N do not commute; therefore,

we must find $\mathbf{v}_1 = (1, 0)^T$, $\mathbf{v}_2 = (1, -2)^T$, $P = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$, $P^{-1} = 1/2 \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ and

$$\mathbf{x}(t) = P \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} P^{-1}\mathbf{x}_0 = 1/2 \begin{bmatrix} 2e^t & e^t - e^{-t} \\ 0 & 2e^{-t} \end{bmatrix} \mathbf{x}_0.$$

2. (a) $\lambda_1 = \lambda_2 = \lambda_3 = 1$; $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix} = S + N$ where S and N

commute and $N^3 = 0$; therefore $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = e^t\,e^{Nt}\,\mathbf{x}_0 = e^t \begin{bmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ 3t + 2t^2 & 2t & 1 \end{bmatrix} \mathbf{x}_0$.

(b) $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 1$ and we must compute the generalized eigenvectors; $\mathbf{v}_1 = (1, 0, 0)^T$, $\mathbf{v}_2 = (0, 1, 0)^T$ satisfying $(A - \lambda_1 I)^2\,\mathbf{v}_2 = \mathbf{0}$, and $\mathbf{v}_3 = (0, 2, 1)^T$; $S = P\,\text{diag}\,[-1, -1, 1]\,P^{-1} =$

$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$, $N = A - S = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, S and N commute, $N^2 = 0$ and $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 =$

$$e^{St}[I + Nt]\,\mathbf{x}_0 = P\,\text{diag}\,[e^{-t}, e^{-t}, e^t]\;P^{-1}\,[I + Nt]\mathbf{x}_0 = \begin{bmatrix} e^{-t} & te^{-t} & -2te^{-t} \\ 0 & e^{-t} & 2(e^t - e^{-t}) \\ 0 & 0 & e^t \end{bmatrix} \mathbf{x}_0.$$

(c) $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 2$ and in this case there is a basis of eigenvectors: $\mathbf{v}_1 = (1, 1, -1)^T$, $\mathbf{v}_2 = (0, 1, 0)^T$, $\mathbf{v}_3 = (0, 0, 1)^T$; $A = P\,\mathrm{diag}\,[1, 2, 2]\,P^{-1}$ and $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 =$

$$P\,\mathrm{diag}\,[e^t, e^{2t}, e^{2t}]P^{-1}\,\mathbf{x}_0 = \begin{bmatrix} e^t & 0 & 0 \\ e^t - e^{2t} & e^{2t} & 0 \\ e^{2t} - e^t & 0 & e^{2t} \end{bmatrix} \mathbf{x}_0.$$

(d) $\lambda_1 = \lambda_2 = \lambda_3 = 2$; $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = S + N$ where S and N commute

and $N^3 = 0$; therefore, $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = e^{2t}\,e^{Nt}\,\mathbf{x}_0 = e^{2t}\begin{bmatrix} 1 & t & t + t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x}_0.$

3. (a) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$; $A = N$ is nilpotent with $A^3 = 0$ and

$$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 1 - t^2 2 & t^2/2 & t \\ t & -t^2/2 & 1 + t^2/2 & t \\ 0 & -t & t & 1 \end{bmatrix}\mathbf{x}_0.$$

(b) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2$; $A = 2I + N$ where $N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $N^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$,

$N^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $N^4 = 0$ and $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = e^{2t}\,e^{Nt}\,\mathbf{x}_0 = e^{2t}\begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ t^2/2 & t & 1 & 0 \\ t^3/6 & t^2 2 & t & 1 \end{bmatrix}\mathbf{x}_0.$

(c) $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 = 10$ and there is a basis of eigenvectors, $\mathbf{v}_1 = (1, -1, 0, 0)^T$,

$\mathbf{v}_2 = (1, 0, -1, 0)^T$, $\mathbf{v}_3 = (1, 0, 0, -1)^T$, $\mathbf{v}_4 = (1, 2, 3, 4)^T$, $P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 4 \end{bmatrix}$,

$P^{-1} = 1/10 \begin{bmatrix} 2 & -8 & 2 & 2 \\ 3 & 3 & -7 & 3 \\ 4 & 4 & 4 & -6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = P\,\text{diag}\,[1, 1, 1, e^{10t}]\,P^{-1} =$

$$1/10 \begin{bmatrix} 9 + e^{10t} & -1 + e^{10t} & -1 + e^{10t} & -1 + e^{10t} \\ -2 + 2e^{10t} & 8 + 2e^{10t} & -2 + 2e^{10t} & -2 + 2e^{10t} \\ -3 + 3e^{10t} & -3 + 3e^{10t} & 7 + 3e^{10t} & -3 + 3e^{10t} \\ -4 + 4e^{10t} & -4 + 4e^{10t} & -4 + 4e^{10t} & 6 + 4e^{10t} \end{bmatrix} \mathbf{x}_0.$$

(d) $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 1 + i$, $\lambda_4 = 1 - i$, $\mathbf{v}_1 = (1, 1, 0, 0)^T$, $\mathbf{v}_2 = (1, -1, 0, 0)$,

$\mathbf{w}_3 = (0, 0, i, 1)$, $P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 =$

$$P \begin{bmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^t\cos t & -e^t\sin t \\ 0 & 0 & e^t\sin t & e^t\cos t \end{bmatrix} P^{-1}\,\mathbf{x}_0 = \begin{bmatrix} \cosh t & \sinh t & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & e^t\cos t & -e^t\sin t \\ 0 & 0 & e^t\sin t & e^t\cos t \end{bmatrix} \mathbf{x}_0.$$

(e) $\lambda_1 = \lambda_2 = 1 + i$ and the eigenvectors $\mathbf{w}_1 = (i, 1, 0, 0)^T$, $\mathbf{w}_2 = (0, 0, i, 1)^T$ lead to

$$P = I,\ A = S = \text{diag}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{x}(t) = e^t \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix} \mathbf{x}_0.$$

(f) $\lambda_1 = \lambda_2 = 1 + i$; the eigenvector $\mathbf{w}_1 = (i, 1, 0, 0)^T$ and the generalized eigenvector

$\mathbf{w}_2 = (i, 1, i, 1)^T$, satisfying $(A - \lambda_1 I)\,\mathbf{w}_2 = \mathbf{w}_1$, lead to $P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

$P^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $S = P \operatorname{diag}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} P^{-1} = \operatorname{diag}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, and

therefore $N = A - S = \begin{bmatrix} 0 & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \\ 0 & 0 \end{bmatrix}$ with $SN = NS$ and $N^2 = 0$; $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 =$

$$e^t P \operatorname{diag}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} P^{-1}\,[I + Nt]\,\mathbf{x}_0 = e^t \begin{bmatrix} \cos t & -\sin t & t\cos t & -t\sin t \\ \sin t & \cos t & t\sin t & t\cos t \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix} \mathbf{x}_0.$$

4. (a) $\lambda_1 = \lambda_2 = 2$, $P_1 = A - 2I = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $r_1(t) = e^{2t}$, $r_2(t) = te^{2t}$, $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 =$

$$[r_1(t)I + r_2(t)P_1]\mathbf{x}_0 = e^{2t}\begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix} \mathbf{x}_0.$$

(b) $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 2$, $r_1(t) = e^t$, $r_2(t) = e^{2t} - e^t$, $r_3(t) = te^{2t} - e^{2t} + e^t$, $P_1 = A - I =$

$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $P_2 = (A - I)(A - 2I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 =$

$$[r_1(t)I + r_2(t)P_1 + r_3(t)P_2]\,\mathbf{x}_0 = \begin{bmatrix} e^t & 0 & 0 \\ e^t - e^{2t} & e^{2t} & 0 \\ -2e^t + (2-t)e^{2t} & te^{2t} & e^{2t} \end{bmatrix} \mathbf{x}_0.$$

(c) $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 2$, $r_1(t) = e^t$, $r_2(t) = e^{2t} - e^t$, $P_1 = A - I$, $P_2 = 0$,

$$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = [r_1(t)I + r_2(t)P_1]\,\mathbf{x}_0 = \begin{bmatrix} e^t & 0 & 0 \\ e^t - e^{2t} & e^{2t} & 0 \\ e^{2t} - e^t & 0 & e^{2t} \end{bmatrix} \mathbf{x}_0.$$

(d) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2$, $r_1(t) = e^{2t}$, $r_2(t) = te^{2t}$, $r_3(t) = t^2 e^{2t}/2$, $r_4(t) = t^3 e^{2t}/6$.

$P_1 = N$, $P_2 = N^2$, $P_3 = N^3$, as in Problem 3(b); thus, $\mathbf{x}(t) = e^{At}\mathbf{x}_0 =$

$$[r_1(t)I + r_2(t)N + r_3(t)N^2 + r_4(t)N^3] = e^{2t}\begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ t^2/2 & t & 1 & 0 \\ t^3/6 & t^2/2 & t & 1 \end{bmatrix}\mathbf{x}_0.$$

PROBLEM SET 1.8

1. (a), (b), (d), (f) and (h) are already in Jordan canonical form.

(c), (e) $\lambda_1 = 1$, $\lambda_2 = -1$ and $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. (g) $\lambda_1 = 2$, $\lambda_2 = 0$ and $J = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

(i) $\lambda_1 = \lambda_2 = 1$, $\delta_1 = 1$, $\delta_2 = 2$, $\nu_1 = 0$, $\nu_2 = 1$ and $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

2. (a), (b), (c), (d) are already in Jordan canonical form.

(e) and (f) $\lambda_1 = \lambda_2 = 1$, $\lambda_2 = -1$, $\delta_1 = 2$ and $J = \text{diag}\,[1, 1, -1]$.

3. (a)

$$\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$\delta_1 = \delta_2 = \delta_3 = \delta_4 = 4$

$\nu_1 = 4$, $\nu_2 = \nu_3 = \nu_4 = 0$

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$\delta_1 = 3$, $\delta_2 = \delta_3 = \delta_4 = 4$

$\nu_1 = 2$, $\nu_2 = 1$, $\nu_3 = \nu_4 = 0$

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$\delta_1 = 2$, $\delta_2 = 3$, $\delta_3 = \delta_4 = 4$

$\nu_1 = 1$, $\nu_2 = 0$, $\nu_3 = 1$, $\nu_4 = 0$

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$\delta_1 = 1$, $\delta_2 = 2$, $\delta_3 = 3$, $\delta_4 = 4$

$\nu_1 = \nu_2 = \nu_3 = 0$, $\nu_4 = 1$

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$\delta_1 = 2$, $\delta_2 = \delta_3 = \delta_4 = 4$

$\nu_1 = \nu_3 = \nu_4 = 0$, $\nu_2 = 2$

(b) $\mathbf{x}(t) = e^{\lambda t}\,\mathbf{x}_0$, $\mathbf{x}(t) = e^{\lambda t} P \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}\,\mathbf{x}_0$. $\mathbf{x}(t) = e^{\lambda t} P \begin{bmatrix} 1 & t & t^2/2 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}\,\mathbf{x}_0$.

$\mathbf{x}(t) = e^{\lambda t} P \begin{bmatrix} 1 & t & t^2/2 & t^3/6 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}\,\mathbf{x}_0$. $\mathbf{x}(t) = e^{\lambda t} P \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}\,\mathbf{x}_0$.

4. (a) $\begin{bmatrix} a_1 & -b_1 & 0 & 0 \\ b_1 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & -b_2 \\ 0 & 0 & b_2 & a_2 \end{bmatrix}$, $\begin{bmatrix} a & -b & 1 & 0 \\ b & a & 0 & 1 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{bmatrix}$, $\begin{bmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}$, $\begin{bmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$.

(b) $\mathbf{x}(t) = P \begin{bmatrix} e^{a_1 t} R_{b_1 t} & 0 \\ 0 & e^{a_2 t} R_{b_2 t} \end{bmatrix} P^{-1}\,\mathbf{x}_0$, $\mathbf{x}(t) = e^{at} P \begin{bmatrix} R_{bt} & tR_{bt} \\ 0 & R_{bt} \end{bmatrix} P^{-1}\,\mathbf{x}_0$.

$\mathbf{x}(t) = P \begin{bmatrix} e^{at} R_{bt} & 0 \\ 0 & \begin{matrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{matrix} \end{bmatrix} P^{-1}\,\mathbf{x}_0$. $\mathbf{x}(t) = P \begin{bmatrix} e^{at} R_{bt} & 0 \\ 0 & \begin{matrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{matrix} \end{bmatrix} P^{-1}\,\mathbf{x}_0$.

5. (a) $\begin{bmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \lambda & & \\ & & & \lambda & \\ & & & & \lambda \end{bmatrix}$, $\delta_1 = 5$

$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & & & \\ & & \lambda & & \\ & & & \lambda & \\ & & & & \lambda \end{bmatrix}$, $\delta_1 = 4$. $\delta_2 = 5$

$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & & \\ & & & \lambda & \\ & & & & \lambda \end{bmatrix}$, $\delta_1 = 3$. $\delta_2 = 4$. $\delta_3 = 5$

$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \lambda & \\ & & & & \lambda \end{bmatrix}$, $\delta_1 = 2$. $\delta_2 = 3$. $\delta_3 = 4$. $\delta_4 = 5$

$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}$, $\delta_1 = 1$, $\delta_2 = 2$. $\delta_3 = 3$. $\delta_4 = 4$. $\delta_5 = 5$

$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & & & \\ & & \lambda & 1 & \\ & & & \lambda & \\ & & & & \lambda \end{bmatrix}$. $\delta_1 = 3$, $\delta_2 = 5$. $\delta_3 = \delta_4 = \delta_5 = 0$

$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}$ $\delta_1 = 2$, $\delta_2 = 4$. $\delta_3 = 5$. $\delta_4 = \delta_5 = 0$.

(b) For example, in the fifth case we have $\mathbf{x}(t) = e^{\lambda t} P \begin{bmatrix} 1 & t & t^2/2 & t^3/6 & t^4/24 \\ 0 & 1 & t & t^2/2 & t^3/6 \\ 0 & 0 & 1 & t & t^2/2 \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} P^{-1} \mathbf{x}_0$.

6. (a) $J = \text{diag}[1, 2, 3]$.

(b) $\lambda_1 = 1, \lambda_2 = \lambda_3 = 2, \delta_1 = 2$, and $J = \text{diag}\,[1, 2, 2]$.

(c) $\lambda_1 = 1, \lambda_2 = \lambda_3 = 2, \delta_1 = 1, \delta_2 = 2$ and $J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

(d) $\lambda_1 = \lambda_2 = \lambda_3 = 2, \delta_1 = 1, \delta_2 = 2, \delta_3 = 3$ and $J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

(e) $J = \text{diag}\,[1, 2, 3, 4]$.

(f) $\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = 2, \delta_1 = 2, \delta_2 = 3$ (for $\lambda = 2$) and $J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

(g) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2, \delta_1 = 2, \delta_2 = 3, \delta_3 = 4$ and as in Problem 3(a), $J = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

(h) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2, \delta_1 = 1, \delta_2 = 2, \delta_3 = 3, \delta_4 = 4$ and see Problem 3(a) solution.

The solutions, which follow from $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = P\,e^{Jt}\,P^{-1}\,\mathbf{x}_0$:

(a) $$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \operatorname{diag}\,[e^{t}, e^{2t}, e^{3t}] \begin{bmatrix} 1/2 & 0 & 0 \\ 1 & 1 & 0 \\ 3/2 & 2 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} e^{t} & 0 & 0 \\ e^{2t} - e^{t} & e^{2t} & 0 \\ \frac{e^{t}}{2} - 2e^{2t} + \frac{3e^{3t}}{2} & 2e^{3t} - 2e^{2t} & e^{3t} \end{bmatrix}.$$

(b) $$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \operatorname{diag}[e^{t}, e^{2t}, e^{2t}] \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} e^{t} & 0 & 0 \\ e^{t} - e^{2t} & e^{2t} & 0 \\ e^{2t} - e^{t} & 0 & e^{2t} \end{bmatrix} \mathbf{x}_0.$$

(c) $$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}_0 =$$

$$\begin{bmatrix} e^{t} & e^{2t} - e^{t} & e^{2t} - e^{t} + te^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \mathbf{x}_0.$$

(d) $$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & te^{2t} & t^2e^{2t}/2 \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}_0 =$$

$$e^{2t} \begin{bmatrix} 1 & t & 2t + t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}_0.$$

(f) $$x(t) = e^{At}x_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{2t} & te^{2t} \\ 0 & 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} x_0 =$$

$$\begin{bmatrix} e^t & 0 & 0 & 0 \\ e^{2t} - e^t & e^{2t} & 0 & 0 \\ e^{2t} - e^t & 0 & e^{2t} & 0 \\ te^{2t} & te^{2t} & 0 & e^{2t} \end{bmatrix} x_0.$$

(h) $$x(t) = e^{At}x_0 = e^{2t} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 12 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & t & t^2/2 & t^3/6 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_0 =$$

$$e^{2t} \begin{bmatrix} 1 & t & 4t + t^2/2 & 3t^2/2 + t^3/6 \\ 0 & 1 & t & t^2/2 - t \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} x_0.$$

7. If $Q = \text{diag}\,[1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{m-1}]$, then $Q^{-1} = \text{diag}\left[1, \frac{1}{\varepsilon}, \frac{1}{\varepsilon^2}, \ldots, \frac{1}{\varepsilon^{m-1}}\right]$ and $Q^{-1}BQ =$

$Q^{-1}(\lambda I + N)Q = \lambda I + Q^{-1}NQ$ where $Q^{-1}NQ =$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\varepsilon} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\varepsilon^2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\varepsilon^{m-1}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon & 0 & \cdots & 0 \\ 0 & 0 & \varepsilon^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \varepsilon^{m-1} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1/\varepsilon & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1/\varepsilon^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1/\varepsilon^{m-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon & 0 & \cdots & 0 \\ 0 & 0 & \varepsilon^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \varepsilon^{m-1} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & \varepsilon & 0 & 0 & \cdots & 0 \\ 0 & 0 & \varepsilon & 0 & \cdots & 0 \\ 0 & 0 & 0 & \varepsilon & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \varepsilon \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \varepsilon N.$$

8. The eigenvalues of a nilpotent matrix are all equal to zero. (This follows from the fact that any nilpotent matrix is linearly equivalent to a matrix with blocks of the form N^k along the diagonal where each N^k has the form of one of the matrices shown on the page following the statement of the theorem in this section.)

9. By the corollary in this section, each coordinate of the solution $\mathbf{x}(t)$ of the initial value problem (4) is a linear combination of functions of the form $t^k e^{at} \cos bt$ or $t^k e^{at} \sin bt$ where k is a non-negative integer and the coefficients depend on the initial conditions $\mathbf{x}_0$. But if all of the eigenvalues of A have a negative real part, then $a = \mathrm{Re}(\lambda) < 0$ in these functions and since for all $a < 0$ and all integers k, $t^k e^{at} \to 0$ as $t \to \infty$ (and since $|\cos bt| \leq 1$ and $|\sin bt| \leq 1$), it follows that for all $\mathbf{x}_0 \in \mathbf{R}^n$ each coordinate of $\mathbf{x}(t)$ approaches zero as $t \to \infty$; i.e., $\mathbf{x}(t) \to 0$ as $t \to \infty$.

10. If the elementary blocks in the Jordan form of A have the form $B = \mathrm{diag}\,[\lambda, \ldots, \lambda]$ or $B = \mathrm{diag}\,[D, \ldots, D]$ where D is a 2 x 2 matrix of the form in the theorem stated in this section, then each coordinate in the solution $\mathbf{x}(t)$ of the initial value problem (4) will be a linear combination of functions of the form $e^{\lambda t}$, $e^{at} \cos bt$ or $e^{at} \sin bt$. Furthermore, if all of the eigenvalues of A have non-positive real part, i.e., if $\lambda \leq 0$ and $a \leq 0$ in the above forms, then each of the coordinates of $\mathbf{x}(t)$ are bounded by constants (depending on $\mathbf{x}_0 \in \mathbf{R}^n$) for all $t \geq 0$ and therefore for each $\mathbf{x}_0 \in \mathbf{R}^n$, there exists a positive constant M such that $|\mathbf{x}(t)| \leq M$ for all $t \geq 0$.

11. Example 4 in Section 1.7 has $|\lambda| = 1$ and yet the functions $t\cos t$ and $t\sin t$ are not bounded as $t \to \infty$ (or as $t \to -\infty$). In particular, the solution with $\mathbf{x}_0 = (1, 0, 0, 0)^T$ has $|\mathbf{x}(t)| = \sqrt{1+t^2+\sin^2 t+t\sin 2t} \geq |t-1|$ and therefore $|\mathbf{x}(t)| \to \infty$ as $t \to \infty$. Also, note that any solution of Example 4 in Section 1.7 with $\mathbf{x}_0 \in \text{Span}\,\{(0, 0, 1, 0)^T, (0, 0, 0, 1)^T\}$ remains bounded for all $t \in \mathbf{R}$.

12. Since this problem is closely related to Problem 5 in Set 9, we shall use the notation and theorems of the next section and do both of these problems at the same time: the corollary in this section tells us that the components of $\mathbf{x}(t)$ are linear combinations of functions of the form $t^k e^{at}\cos bt$ or $t^k e^{at}\sin bt$ with $\lambda = a + ib$ and $0 \leq k \leq n-1$.

(a) This case occurs iff $\mathbf{x}_0 \in E^s \sim \{\mathbf{0}\}$.

(b) This case occurs iff $\mathbf{x}_0 \in E^u \sim \{\mathbf{0}\}$.

(c) This case occurs if $\mathbf{x}_0 \in E^c \sim \{\mathbf{0}\}$ and A is semisimple. (It may also occur if $\mathbf{x}_0 \in E^c \sim \{\mathbf{0}\}$ even if A is not semisimple as in Example 4 in Section 1.7.) That $|\mathbf{x}(t)| \geq m$ follows from the fact that $\mathbf{x}(t)$ is a periodic solution which does not intersect the critical point at the origin.

(d) This case occurs if $E^s \neq \{\mathbf{0}\}$, $E^u \neq \{\mathbf{0}\}$ and $\mathbf{x}_0 \in E^u \oplus E^s \oplus E^c \sim (E^u \cup E^s \cup E^c)$. (It may also occur for certain $\mathbf{x}_0 \in E^c \sim \{\mathbf{0}\}$ as in Example 4 in Section 1.7; cf. Problem 11 above.)

(e) This case occurs if $E^u \neq \{\mathbf{0}\}$, $E^c \neq \{\mathbf{0}\}$ and $\mathbf{x}_0 \in E^u \oplus E^c \sim (E^u \cup E^c)$.

(f) This case occurs if $E^s \neq \{\mathbf{0}\}$, $E^c \neq \{\mathbf{0}\}$ and $\mathbf{x}_0 \in E^s \oplus E^c \sim (E^s \cup E^c)$.

Furthermore, these are the only possible types of behavior that can occur as $t \to \pm\infty$ according to the corollary in this section.

PROBLEM SET 1.9

1. (a) $E^s = \text{Span}\{(0, 1)^T\}$, $E^u = \text{Span}\{(1, 0)^T\}$, $E^c = \{\mathbf{0}\}$.

   (b) $E^s = E^u = \{\mathbf{0}\}$, $E^c = \mathbf{R}^2$.

   (c) $E^s = E^c = \{\mathbf{0}\}$, $E^u = \mathbf{R}^2$.

   (d) $E^s = \text{Span}\{(1, 0)^T\}$, $E^u = \text{Span}\{(1, -1)^T\}$, $E^c = \{\mathbf{0}\}$.

   (e, f) $E^s = \text{Span}\{(1\ -1)^T\}$, $E^u = \text{Span}\{(1, 0)^T\}$, $E^c = \{\mathbf{0}\}$.

   (g) $E^s = \text{Span}\{(0, 1)^T\}$, $E^u = \{\mathbf{0}\}$, $E^c = \text{Span}\{(1, 0)^T\}$.

   (h) $E^s = E^u = \{\mathbf{0}\}$, $E^c = \mathbf{R}^2$.

   (i) $E^s = \mathbf{R}^2$, $E^s = E^c = \{\mathbf{0}\}$.

   The flow e^{At} is hyperbolic exactly when $E^c = \{\mathbf{0}\}$.

2. (a) $E^s = \text{Span}\{(1, 0, 0)^T, (0, 1, 0)^T\}$, $E^u = \text{Span}\{(0, 0, 1)^T\}$, $E^c = \{\mathbf{0}\}$.

   (b) $E^s = \text{Span}\{(0, 0, 1)^T\}$, $E^u = \{\mathbf{0}\}$, $E^c = \text{Span}\{(1, 0, 0)^T, (0, 1, 0)^T\}$.

   (c) $E^s = \text{Span}\{(1, 0, 0)^T, (0, 0, 1)^T\}$, $E^u = \text{Span}\{1, -1, 0)^T\}$, $E^c = \{\mathbf{0}\}$.

   (d) $E^s = \text{Span}\{(1, -1, 0)^T, (0, 0, 1)^T\}$, $E^u = \text{Span}\{(1, 0, 0)^T\}$, $E^c = \{\mathbf{0}\}$.

   The flow is hyperbolic in (a, c, d).

3. $\lambda = \pm 2i$, $\lambda_3 = 6$, $\mathbf{w}_1 = \mathbf{u}_1 + i\mathbf{v}_1 = (10, 0, -3)^T + i(0, 10, -1)^T$, $\mathbf{v}_3 = (0, 0, 1)^T$.

   $E^s = \{\mathbf{0}\}$, $E^u = \text{Span}\{(0, 0, 1)^T\}$, $E^c = \text{Span}\{(0, 10, -1)^T, (10, 0, -3)^T\}$.

$$\mathbf{x}(t) = \frac{1}{10}\begin{bmatrix} 0 & 10 & 0 \\ 10 & 0 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} \cos 2t & -\sin 2t & 0 \\ \sin 2t & \cos 2t & 0 \\ 0 & 0 & e^{6t} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} \mathbf{x}_0 =$$

$$\frac{1}{10}\begin{bmatrix} 10\cos 2t & 10\sin 2t & 0 \\ -10\sin 2t & 10\cos 2t & 0 \\ \sin 2t - 3\cos 2t + 3e^{6t} & -\cos 2t - 3\sin 2t + e^{6t} & e^{6t} \end{bmatrix} \mathbf{x}_0.$$

For $\mathbf{x}_0 = (0, 0, c)^T \in E^u$, $\mathbf{x}(t) = (0, 0, e^{6t}c)^T \in E^u$; for $\mathbf{x}_0 \in E^c$, i.e., for $\mathbf{x}_0 =$ $(10a, 10b, -3a - b)^T$, $\mathbf{x}(t) = (10(a\cos 2t + b\sin 2t), 10(b\cos 2t - a\sin 2t),$ $-3(a\cos 2t + b\sin 2t) - (b\cos 2t - a\sin 2t))^T \in E^c$; and for $\mathbf{x}_0 = \mathbf{0} \in E^s$, $\mathbf{x}(t) = \mathbf{0} \in E^s$.

4. (a) $E^s = \text{Span}\,\{(1, 0, 0.)^T, (0, 1, 0)^T\}$, $E^u = \text{Span}\,\{(0, 2, 1)^T\}$, $E^c = \{\mathbf{0}\}$.

(b) $E^s = E^c = \{\mathbf{0}\}$, $E^u = \mathbf{R}^3$.

5. See Problem 12 in Set 8.

6. If $L : ax_1 + bx_2 = 0$ is an invariant line for the system (1), then for $\mathbf{x}_0 \in L$; $e^{At}\,\mathbf{x}_0 \in L$ for all $t \in \mathbf{R}$. But $\mathbf{x}_0 = (x_1, x_2)^T \in L$ and $\mathbf{x}_0 \neq \mathbf{0}$ implies that $\mathbf{x}_0 = k_1(-b, a)^T$ with $k_1 \neq 0$. And then $e^{At}\,\mathbf{x}_0 \in L$ for all $t \in \mathbf{R}$ implies that for all $t \in \mathbf{R}$ $e^{At}\,k_1(-b, a)^T = k_2(-b, a)^T$, and in particular that $e^A\mathbf{v} = k\mathbf{v}$ with $k = k_2/k_1$ and $\mathbf{v} = (-b, a)^T$. As in Section 1.5, if (1) has an invariant line then $A = PBP^{-1}$ and either $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ or $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. In the first case, if $\lambda \neq \mu$ it follows that either $k = e^\lambda$ and $P^{-1}\mathbf{v} = (1, 0)^T$ is an eigenvector of B, i.e., $\mathbf{v}$ is an eigenvector of A, or $k = e^\mu$ and $P^{-1}\mathbf{v} = (0, 1)^T$ is an eigenvector of B, i.e., $\mathbf{v}$ is an eigenvector of A. Also, in the first case if $\lambda = \mu$, then any vector $\mathbf{v} \in \mathbf{R}^2$ is an eigenvector of A and, in particular, $\mathbf{v} = (-b, a)^T$ is an eigenvector of A. In the second case $k = e^\lambda$ and $P^{-1}\mathbf{v} = (1, 0)^T$ is an eigenvector of B, i.e., $\mathbf{v}$ is an eigenvector of A and we are done. (The converse of Problem 6, that if $\mathbf{v} = (v_1, v_2)^T$ is an eigenvector of A, then $v_2x_1 - v_1x_2 = 0$ is an invariant line of (1) follows immediately from Problem 6 in Set 3.)

PROBLEM SET 1.10

1. Let $\Phi(t)$ be a fundamental matrix solution of (2) and let $\mathbf{x}(t) = \Phi(t)\mathbf{c}(t)$. Then $\mathbf{c}(0) = \Phi^{-1}(0)\mathbf{x}_0$ and $\dot{\mathbf{x}}(t) = \dot{\Phi}(t)\mathbf{c}(t) + \Phi(t)\dot{\mathbf{c}}(t) = A\Phi(t)\mathbf{c}(t) + \Phi(t)\dot{\mathbf{c}}(t)$ while $A\mathbf{x}(t) + \mathbf{b}(t) = A\Phi(t)\mathbf{c}(t) + \mathbf{b}(t)$. It then follows from (1) that $\Phi(t)\dot{\mathbf{c}}(t) = \mathbf{b}(t)$, i.e., that $\mathbf{c}(t) = \mathbf{c}(0) + \int_0^t \Phi^{-1}(\tau)\mathbf{b}(\tau)d\tau = \Phi^{-1}(0)\mathbf{x}_0 + \int_0^t \Phi^{-1}(\tau)\mathbf{b}(\tau)d\tau$. Thus $\mathbf{x}(t) = \Phi(t)\mathbf{c}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{x}_0 + \Phi(t)\int_0^t \Phi^{-1}(\tau)\mathbf{b}(\tau)d\tau$ which is equation (3).

2. $\lambda_1 = 1, \lambda_2 = -1$, $\mathbf{v}_1 = (1, 0)^T$, $\mathbf{v}_2 = (1, -2)^T$ and a fundamental matrix $\Phi(t)$ with $\Phi(0) = I$ is given by $\Phi(t) = e^{At} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix} = \begin{bmatrix} e^t & (e^t - e^{-t})/2 \\ 0 & e^{-t} \end{bmatrix}$.

Note that $\Phi^{-1}(t) = \Phi(-t)$ and then

$$\mathbf{x}(t) = \begin{bmatrix} e^t & \dfrac{(e^t - e^{-t})}{2} \\ 0 & e^{-t} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{bmatrix} e^t & \dfrac{(e^t - e^{-t})}{2} \\ 0 & e^{-t} \end{bmatrix} \int_0^t \begin{bmatrix} e^{-\tau} & \dfrac{(e^{-\tau} - e^{\tau})}{2} \\ 0 & e^{\tau} \end{bmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} d\tau =$$

$$\begin{pmatrix} -t - 2 + \dfrac{5}{2} e^t + \dfrac{1}{2} e^{-t} \\ 1 - e^{-t} \end{pmatrix}.$$

3. $\dot{\Phi}(t) = \begin{bmatrix} -2e^{-2t}\cos t - e^{-2t}\sin t & -\cos t \\ -2e^{-2t}\sin t + e^{-2t}\cos t & -\sin t \end{bmatrix} = A(t)\Phi(t)$, $\Phi^{-1}(t) = e^{2t}\begin{bmatrix} \cos t & \sin t \\ -e^{-2t}\sin t & e^{-2t}\cos t \end{bmatrix}$,

$\Phi^{-1}(0) = I$, and $\mathbf{x}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{x}_0 + \Phi(t)\int_0^t e^{2\tau}\begin{bmatrix} \cos\tau & \sin\tau \\ -2e^{-2\tau}\sin\tau & e^{-2\tau}\cos\tau \end{bmatrix}\begin{pmatrix} 1 \\ e^{-2\tau} \end{pmatrix} d\tau =$

$$\Phi(t)\left[\mathbf{x}_0 + \begin{pmatrix} \dfrac{e^{2t}}{5}(2\cos t + \sin t) - \cos t + \dfrac{3}{5} \\ \dfrac{-e^{-2t}}{5}(2\cos t - \sin t) + \cos t - \dfrac{3}{5} \end{pmatrix}\right] =$$

$$\Phi(t)\mathbf{x}_0 + \frac{1}{5}\begin{pmatrix} 2\cos^2 t - 4\sin t\cos t + 3\sin t + e^{-2t}(-5\cos^2 t + 2\sin t\cos t - \sin^2 t + 3\cos t) \\ \sin^2 t + 2\sin t\cos t + 5\cos^2 t - 3\cos t + e^{-2t}(-2\cos^2 t - 4\sin t\cos t + 3\sin t) \end{pmatrix}.$$

2. NONLINEAR SYSTEMS: LOCAL THEORY

PROBLEM SET 2.1

1. $Df(\mathbf{x}) = \begin{bmatrix} 1+x_2^2 & 2x_1x_2 \\ 2x_1 & -1+2x_2 \end{bmatrix}$, $Df(\mathbf{0}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $Df(0,1) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

   $D^2f(0, 1)(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} 2x_1y_2 + 2x_2y_1 \\ 2x_1y_1 + 2x_2y_2 \end{pmatrix}$.

2. (a) $E = \mathbf{R}^2 \sim \{\mathbf{0}\}$. (b) $E = \{\mathbf{x} \in \mathbf{R}^2 \mid x_1 > -1, x_2 > -2, x_1 \neq 1\} \sim \{\mathbf{0}\}$.

3. $x(t) = \begin{Bmatrix} t^2/4, t \geq 0 \\ -t^2/4, t \leq 0 \end{Bmatrix}$, $x(t) = \begin{Bmatrix} 0, t \geq 0 \\ -t^2/4, t \leq 0 \end{Bmatrix}$, $x(t) = \begin{Bmatrix} t^2/4, t \geq 0 \\ 0, t \leq 0 \end{Bmatrix}$, $x(t) = 0$.

4. $x(t) = 2/\sqrt{1-8t}$ for $-\infty < t < 1/8$ and $x(t) \to \infty$ as $t \to 1/8^-$.

5. $x(t) = \sqrt{t}$ is a solution on $(0, \infty)$ but not on $[0, \infty)$ since $x'(t) = 1/2\sqrt{t}$ is undefined at $t = 0$.

6. $\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| = \max\limits_{|a|=1} \sqrt{[(x_1 - y_1)a_1 + (x_2 - y_2)a_2]^2 + [(y_2 - x_2)a_1 + (x_1 - y_1)a_2]^2}$.

   Thus, if $|\mathbf{x} - \mathbf{y}| < \delta$, then $|x_1 - y_1| < \delta$, $|x_2 - y_2| < \delta$ and therefore

   $\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| < \max\limits_{|a|=1} \delta\sqrt{(a_1 + a_2)^2 + (a_1 + a_2)^2} \leq 2\delta = \varepsilon$ if $\delta = \varepsilon/2$.

PROBLEM SET 2.2

1. (a) $u_1(t) = 1 + t$, $u_2(t) = 1 + t + t^2 + t^3/3$, $u_3(t) = 1 + t + t^2 + t^3 + 2t^4/3 + t^5/3 + t^6/9 + t^7/63$.

   Mathematical induction: $u_1(t) = 1 + t$, $u_2(t) = 1 + t + t^2 + 0(t^3)$ and for $n \geq 1$, assuming $u_n(t) = 1 + t + t^2 + \cdots + t^n + 0(t^{n+1})$ we find that $u_{n+1}(t) = 1 + \int_0^t [1 + s + s^2 + \cdots + s^n + 0(s^{n+1})]^2 ds = 1 + \int_0^t [1 + 2s + 3s^2 + \cdots + (n+1)s^n + 0(s^{n+1})]ds = 1 + t + t^2 + \cdots + t^{n+1} + 0(t^{n+2})$. QED.

(b) By separating variables and integrating we find that $x(t) = 1/(c - t)$ and the initial condition implies that $c = 1$. For $x(t) = (1 - t)^{-1}$, we have that $\dot{x}(t) = (1 - t)^{-2} = x^2(t)$ for $t \neq 0$; and since $x(0) = 1$, $t \in (-\infty, 1)$, and this function is a solution of the IVP in part (a) according to Definition 1. The Taylor series for $x(t) = 1/(1 - t) = 1 + t + \cdots + t^n + \cdots$, which agrees with the first $(n + 1)$-terms in $u_n(t)$ found in part (a).

(c) $\dot{x}(t) = (3t)^{-2/3} = 1/x^2(t)$ for all $t \neq 0$; hence the function $x(t) = (3t)^{1/3}$ is a solution of the given differential equation on the interval $(-\infty, 0)$ or on the interval $(0, \infty)$. Clearly this function satisfies $x(1/3) = 1$, $1/3 \in (0, \infty)$ and hence $x(t) = (3t)^{1/3}$ is a solution of the given IVP on the interval $(0, \infty)$ according to Definition 1.

2. $\mathbf{u}_0(t) = \mathbf{x}_0$, $\mathbf{u}_1(t) = \mathbf{x}_0 + A\mathbf{x}_0$, $\cdots$, $\mathbf{u}_k(t) = (I + A + \cdots + A^k/k!)\mathbf{x}_0$ and $\lim_{k\to\infty} \mathbf{u}_k(t) = e^{At}\mathbf{x}_0$ absolutely and uniformly on any interval $[0, t_0]$.

5. By the lemma in this section, f is locally Lipschitz in E. Therefore, given $\mathbf{x}_0 \in E$, there exists a $K_0 > 0$ and an $\varepsilon > 0$ such that $N_\varepsilon(\mathbf{x}_0) \subset E$ and for all $\mathbf{x}, \mathbf{y} \in N_\varepsilon(\mathbf{x}_0)$, $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K_0 |\mathbf{x} - \mathbf{y}|$. Next, $T \circ \mathbf{u}(t)$ is continuous at $t = 0$. Therefore for $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|t| < \delta$ then $|T \circ \mathbf{u}(t) - T \circ \mathbf{u}(0)| = |T \circ \mathbf{u}(t) - \mathbf{x}_0| < \varepsilon$. Choose $a > 0$ such that $a < \min(\delta, 1/K_0)$. Then for $I = [-a, a]$, $t \in I$ and $\mathbf{u}, \mathbf{v} \in V \equiv \{\mathbf{u} \in C(I) \mid \|\mathbf{u} - \mathbf{x}_0\| \leq \varepsilon\}$.
$|T \circ \mathbf{u}(t) - T \circ \mathbf{v}(t)| = \left|\int_0^t [\mathbf{f}(\mathbf{u}(s)) - \mathbf{f}(\mathbf{v}(s))]ds\right| \leq \int_0^t |\mathbf{f}(\mathbf{u}(s)) - \mathbf{f}(\mathbf{v}(s))| ds \leq c\|\mathbf{u} - \mathbf{v}\|$ where $c = K_0 a < 1$. Thus, by the contraction mapping principle, there exists a unique $\mathbf{u}(t) \in V \subset C(I)$ such that $T \circ \mathbf{u}(t) = \mathbf{u}(t)$ for all $t \in I$.

6. If $\mathbf{x}(t)$ is a continuous function on I that satisfies the integral equation, then $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(t) = \frac{d}{dt}\int_0^t \mathbf{f}(\mathbf{x}(s))ds = \mathbf{f}(\mathbf{x}(t))$ for all $t \in I$ by the fundamental theorem of calculus since $\mathbf{f}(\mathbf{x}(t)) \in C(I)$; and therefore $\mathbf{x}(t)$ is differentiable and it satisfies the initial value problem (2) for all $t \in I$. Conversely, if $\mathbf{x}(t)$ is a solution of the initial value problem (2) for all $t \in I$, then $\mathbf{x}(t)$ is differentiable and hence continuous on I and $\mathbf{x}(t) \in E$ for all $t \in I$; therefore,

$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$ implies that $\mathbf{x}(t) = \int_0^t \mathbf{f}(\mathbf{x}(s))ds + \mathbf{c}$ for all $t \in I$ and clearly $c = \mathbf{x}(0) = \mathbf{x}_0$. Thus, $\mathbf{x}(t)$ satisfies the integral equation for all $t \in I$.

7. $\ddot{\mathbf{x}}(t) = \frac{d}{dt}\left[\mathbf{f}(\mathbf{x}(t))\right] = D\mathbf{f}[\mathbf{x}(t)]\dot{\mathbf{x}}(t) = D\mathbf{f}[\mathbf{x}(t)]\mathbf{f}(\mathbf{x}(t)) \in C(I)$ by the chain rule since $\mathbf{x}(t) \in E$, $D\mathbf{f}[\mathbf{x}(t)]$ and $\mathbf{f}(\mathbf{x}(t))$ are continuous for all $t \in I$.

8. Since a continuous function on a compact set is bounded, $D\mathbf{f}$ is bounded on E. It then follows immediately from Theorem 9.19 in [R] that $\mathbf{f}$ satisfies a Lipschitz condition on E.

9. Suppose that there is a constant $K_0 > 0$ such that for all $\mathbf{x}, \mathbf{y} \in E$, $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \le K_0 |\mathbf{x} - \mathbf{y}|$. Then, given $\varepsilon > 0$, choose $\delta = \varepsilon/K_0 > 0$ to get that for $\mathbf{x}, \mathbf{y} \in E$ with $|\mathbf{x} - \mathbf{y}| < \delta$, $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \le K_0 |\mathbf{x} - \mathbf{y}| < K_0\delta = \varepsilon$. Therefore, $\mathbf{f}$ is uniformly continuous on E.

10.(a) Follow the hint for $\delta < 1$; and for $\delta \ge 1$, choose $x = 1$ and $y = 1/3$ to show that $|f(x) - f(y)| = 2 > 1 = \varepsilon$.

(b) Use the result of part (a) and Problem 9 to show that $f(x) = 1/x$ does not satisfy a Lipschitz condition on $(0, 1)$.

11. If $\mathbf{f}$ is differentiable at $\mathbf{x}_0$, then there exists a linear transformation $D\mathbf{f}(\mathbf{x}_0)$ such that given $\varepsilon = 1$, there is a $\delta > 0$ such that for $|\mathbf{x} - \mathbf{x}_0| < \delta$, $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0) - D\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)| \le |\mathbf{x} - \mathbf{x}_0|$. Thus, for $K_0 = 1 + \|D\mathbf{f}(\mathbf{x}_0)\|$, we obtain the desired result.

PROBLEM SET 2.3

1. The initial value problem has the solution $\mathbf{u}(t, \mathbf{y}) = e^{At}\mathbf{y}$. Thus, $\Phi(t) = \frac{\partial \mathbf{u}}{\partial \mathbf{y}}(t, \mathbf{y}) = e^{At}$ which is the unique fundamental matrix satisfying $\dot{\Phi} = A\Phi$ and $\Phi(0) = I$.

2. (a) $u_1(t, y) = y_1 e^{-t}$, $u_2(t, y) = -y_1^2 e^{-2t} + (y_1^2 + y_2)e^{-t}$ and $u_3(t, y) = (-y_1^2/3)e^{-2t} + (y_1^2/3 + y_3)e^{t}$.

$$\Phi(t) = \frac{\partial \mathbf{u}}{\partial \mathbf{y}}(t, \mathbf{y}) = \begin{bmatrix} e^{-t} & 0 & 0 \\ -2y_1e^{-2t} + 2y_1e^{-t} & e^{-t} & 0 \\ -\frac{2}{3}y_1e^{-2t} + \frac{2}{3}y_1e^{t} & 0 & e^{t} \end{bmatrix}, \Phi(0) = I \text{ and}$$

$$\dot{\Phi}(t) = \begin{bmatrix} -e^{-t} & 0 & 0 \\ 4y_1e^{-2t} - 2y_1e^{-t} & -e^{-t} & 0 \\ \frac{4}{3}y_1e^{-2t} + \frac{2}{3}y_1e^{t} & 0 & e^{t} \end{bmatrix} = Df[\mathbf{u}(t, \mathbf{y})]\Phi(t) = \begin{bmatrix} -1 & 0 & 0 \\ 2y_1e^{-t} & -1 & 0 \\ 2y_1e^{-t} & 0 & 1 \end{bmatrix} \Phi(t).$$

(b) $\Phi(t) = \begin{bmatrix} (1-y_1t)^{-2} & 0 \\ (1-e^{t})/y_1^2 & e^{t} \end{bmatrix}$ etc.

5. By the corollary in this section, it follows from Liouville's Theorem that $\det \frac{\partial \mathbf{u}}{\partial \mathbf{y}}(t, \mathbf{x}_0) = \exp\int_0^t \text{trace } Df\,[\mathbf{u}(s, \mathbf{x}_0)]\,ds = \exp\int_0^t \nabla\cdot \mathbf{f}(\mathbf{u}(s, \mathbf{x}_0))ds$ since trace $Df = \nabla\cdot\mathbf{f}$.

6. From vector calculus, i.e., from the hint, it follows that $\mathbf{y} = \mathbf{u}(t, \mathbf{y}_0)$ is volume preserving iff $J(\mathbf{x}) = \det \frac{\partial \mathbf{u}}{\partial \mathbf{x}}(t, \mathbf{x}) = 1$ for all $t \in [0, a]$. But, from Problem 5, this follows iff $\int_0^t \nabla\cdot\mathbf{f}(\mathbf{u}(s, \mathbf{y}_0))ds = 0$ for all $t \in [0, a]$ and $\mathbf{y}_0 \in E$; and by continuity, this follows iff $\nabla\cdot\mathbf{f}(\mathbf{x}) = 0$ for all $\mathbf{x} \in E$.

PROBLEM SET 2.4

1. (a) $x(t) = x_0/(1 - x_0t)$; $(\alpha, \beta) = (-\infty, 1/x_0)$ for $x_0 > 0$ and $x(t) \to \infty$ as $t \to (1/x_0)^-$; $(\alpha, \beta) = (-\infty, \infty)$ for $x_0 = 0$; and $(\alpha, \beta) = (1/x_0, \infty)$ for $x_0 < 0$ and $x(t) \to -\infty$ as $t \to (1/x_0)^+$.

   (b) $(\alpha, \beta) = (-1, 1)$ and $x(t) = \sin^{-1}(t) \to \mp\pi/2 \in \dot{E}$ as $t \to \alpha^+$ or as $t \to \beta^-$ where $E = (-\pi/2, \pi/2)$.

   (c) $x(t) = -2\tanh(2t)$ and $(\alpha, \beta) = (-\infty, \infty)$.

(d) $x(t) = |x_0|/(1 - 2x_0^2 t)^{1/2}$, $(\alpha, \beta) = (-\infty, 1/2x_0^2)$ and $x(t) \to \infty$ as $t \to (1/2x_0^2)^-$.

(e) $x_1(t) = y_1/(1 - y_1 t)$, $x_2(t) = (y_2 - 1 + 1/y_1)e^t + (1 + t - 1/y_1)$, $(\alpha, \beta) = (-\infty, 1/y_1)$ and $|\mathbf{x}(t)| \to \infty$ as $t \to (1/y_1)^-$.

2. (a) $x_1(t) = (1 - t)^{-1}$, $x_2(t) = t + e^t$, $(\alpha, \beta) = (-\infty, 1)$ and $|\mathbf{x}(t)| \to \infty$ as $t \to 1^-$.

(b) $x_1(t) = \sqrt{1+t}$, $x_2(t) = (1 - t)^{-1}$, $(\alpha, \beta) = (-1, 1)$, $\mathbf{x}(t) \to (0, .5)^T \in \dot{E}$ as $t \to (-1)^+$, where $E = \{x_1 > 0\}$, and $|\mathbf{x}(t)| \to \infty$ as $t \to 1^-$.

(c) $x_1(t) = \sqrt{1+t}$, $x_2(t) = (2/3)(1 + t)^{3/2} + 1/3$, $(\alpha, \beta) = (-1, \infty)$ and $\mathbf{x}(t) \to (0, 1/3)^T \in \dot{E}$ as $t \to (-1)^+$, where $E = \{x_1 > 0\}$.

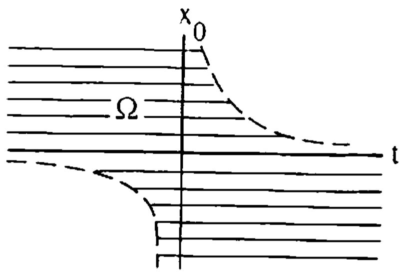
3. Asssume $\beta < \infty$. If $\lim_{t\to\infty^-} \mathbf{x}(t)$ does not exist, then there exists a sequence $t_n \to \beta^-$ such that $\{\mathbf{x}(t_n)\}$ is not Cauchy; i.e., there exists an $\varepsilon > 0$ such that for all integers N, there exist integers $n > m \geq N$ such that $|\mathbf{x}(t_n) - \mathbf{x}(t_m)| \geq \varepsilon$. Thus, for N = 1, there exists integers $n_1 > m_1 \geq 1$ such that $|\mathbf{x}(t_{n_1}) - \mathbf{x}(t_{m_1})| \geq \varepsilon$; for $N = n_1$, there exist integers $n_2 > m_2 \geq n_1$ such that $|\mathbf{x}(t_{n_2}) - \mathbf{x}(t_{m_2})| \geq \varepsilon$; $\cdots$ for $N = n_j$, there exist integers $n_{j+1} > m_{j+1} \geq n_j$ such that $|\mathbf{x}(t_{n_j}) - \mathbf{x}(t_{m_j})| \geq \varepsilon$. Hence, the arc length of $\Gamma_+ \geq \sum_{n=1}^{\infty} |\mathbf{x}(t_{n+1}) - \mathbf{x}(t_n)| \geq \sum_{j=1}^{\infty} |\mathbf{x}(t_{n_j}) - \mathbf{x}(t_{m_j})| \geq \sum_{j=1}^{\infty} \varepsilon = \infty$. Hence if $\beta < \infty$ and the arc length of Γ_+ is finite, it follows that $\lim_{t\to\beta^-} \mathbf{x}(t)$ exists.

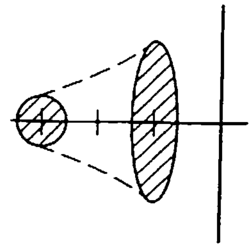
4. In cylindrical coordinates $\dot{r} = 0$, $\dot{\theta} = r^2/x_3^2 = 1/x_3^2$ and $\dot{x}_3 = 1$. Thus, $r = 1$, $x_3(t) = t + 1/\pi$ and $\theta(t) = -(t + 1/\pi)^{-1}$. $(\alpha, \beta) = (-1/\pi, \infty)$, and $\lim \mathbf{x}(t)$ as $t \to (-1/\pi)^+$ does not exist (Γ spirals down toward the unit circle in the x_1, x_2 plane as $t \to (-1/\pi)^+$); also, Γ_+ and Γ_- both have infinite arc length (cf. Problem 3).

5. Suppose $\lim_{t\to\beta^-} \mathbf{x}(t) = \mathbf{x}_1 \in E$. Then since E is open, there is an $\varepsilon > 0$ such that $N_{2\varepsilon}(\mathbf{x}_1) \subset E$ and $\overline{N_\varepsilon(\mathbf{x}_1)} \subset E$. Assume that $\beta < \infty$. Then there is a $\delta > 0$ such that for $|t - \beta| < \delta$, $|\mathbf{x}(t) - \mathbf{x}_1| < \varepsilon$. Since $\mathbf{x}(t)$ is continuous and $[0, \beta - \delta]$ is a compact set, $K = \{\mathbf{y} \in \mathbf{R}^n \mid \mathbf{y} = \mathbf{x}(t), t \in [0, \beta - \delta]\} \cup \{\mathbf{y} \in \mathbf{R}^n \mid |\mathbf{y} - \mathbf{x}_1| \leq \varepsilon\}$ is a compact subset of E; furthermore,

$\Gamma_+ \subset K$. Thus, by Corollary 2, β is not finite; i.e., $\beta = \infty$. Next, we show that $\mathbf{f}(\mathbf{x}_1) = \mathbf{0}$. Suppose that $\mathbf{f}(\mathbf{x}_1) \neq 0$, say $|\mathbf{f}(\mathbf{x}_1)| = \delta > 0$. Then by the continuity of $\mathbf{f}$, there exists an $\varepsilon > 0$ such that $|\mathbf{x} - \mathbf{x}_1| < \varepsilon$ implies that $\mathbf{x} \in E$ and $|\mathbf{f}(\mathbf{x})| \geq \delta/2$. Since $\mathbf{x}(t) \to \mathbf{x}_1$ and $\dot{\mathbf{x}}(t) \to \mathbf{v}_1 \equiv \mathbf{f}(\mathbf{x}_1)$ as $t \to \infty$, it follows that for this $\varepsilon > 0$, there exists a $t_0 \geq 0$ such that for all $t \geq t_0$, $|\mathbf{x}(t) - \mathbf{x}_1| < \varepsilon$ and $|\dot{\mathbf{x}}(t) - \mathbf{v}_1| < \varepsilon$, i.e., for all $t \geq t_0$, $|\dot{\mathbf{x}}(t)| = |\mathbf{f}(\mathbf{x}(t))| \geq \delta/2$ and $|\mathbf{v}_1 \cdot \dot{\mathbf{x}}(t)| = |\mathbf{v}_1|\,|\dot{\mathbf{x}}(t)| \cdot |\cos\theta_1(t)| \geq |\mathbf{v}_1|\,\delta/4$ where $\theta_1(t)$ is the angle between $\dot{\mathbf{x}}(t)$ and $\mathbf{v}_1$ and $|\cos\theta_1(t)| \geq 1/2$ for all $t \geq t_0$. Then by the mean value theorem, for all $t > t_0$, there is a $\tilde{t} \in (t_0, t)$ such that $\mathbf{v}_1 \cdot \mathbf{x}(t) - \mathbf{v}_1 \cdot \mathbf{x}(t_0) = (t - t_0)\mathbf{v}_1 \cdot \dot{\mathbf{x}}(\tilde{t})$; thus, $|\mathbf{v}_1|\,|\mathbf{x}(t) - \mathbf{x}(t_0)| \geq |\mathbf{v}_1 \cdot [\mathbf{x}(t) - \mathbf{x}(t_0)]| = |t - t_0|\,|\mathbf{v}_1 \cdot \dot{\mathbf{x}}(\tilde{t})| \geq |t - t_0|\,|\mathbf{v}_1|\,\delta/4$ and therefore $|\mathbf{x}(t) - \mathbf{x}(t_0)| \geq |t - t_0|\,\delta/4 \geq 2\varepsilon$ for $t \geq t_0 + 8\varepsilon/\delta$. Since $|\mathbf{x}(t_0)| < \varepsilon$, this implies that for $t \geq t_0 + 8\varepsilon/\delta$, $|\mathbf{x}(t) - \mathbf{x}_1| \geq |\mathbf{x}(t) - \mathbf{x}(t_0)| - |\mathbf{x}(t_0) - \mathbf{x}_1| \geq 2\varepsilon - \varepsilon = \varepsilon$, a contradiction since $|\mathbf{x}(t) - \mathbf{x}_1| < \varepsilon$ for all $t \geq t_0$. Thus, $\mathbf{f}(\mathbf{x}_1) = \mathbf{0}$; and $\mathbf{x}_1$ is an equilibrium point of (1), i.e., $\mathbf{x}(t) = \mathbf{x}_1$ is the solution of (1) satisfying the initial condition $\mathbf{x}(0) = \mathbf{x}_1$.

PROBLEM SET 2.5





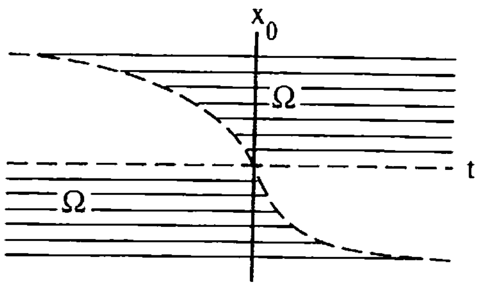
1. $x(t) = x_0/(1 - x_0 t)$.

3. $\phi_t(\mathbf{x}) = \mathrm{diag}[e^{-t}, e^{2t}]\mathbf{x}$.

5. $\phi_t(\mathbf{y}) = (y_1 e^{-t}, -y_1^2 e^{-2t}/4 + (4y_2 + y_1^2)e^{2t}/4)^T$. If $y_2 = -y_1^2/4$, then $\phi_t(\mathbf{y}) = (y_1 e^{-t}, -y_1^2 e^{-2t}/4) \in S$.

6. $\phi_t(\mathbf{y}) = (y_1 e^{-t}, -y_1^2 e^{-2t} + y_2 e^{-t} + y_1^2 e^{-t}, -y_1^2 e^{-2t}/3 + (3y_3 + y_1^2)e^{t}/3).^T$ If $y_3 = -y_1^2/3$, then $\phi_t(\mathbf{y}) = (y_1 e^{-t}, -y_1^2 e^{-2t} + y_2 e^{-t} + y_1^2 e^{-t}, -y_1^2 e^{-2t}/3) \in S$.

7. $\phi_t(x_0) = (3t + x_0^3)^{1/3}$. For $x_0 > 0$, $(\alpha, \beta) = (-x_0^3/3, \infty)$ and $\phi_t(x_0) \to 0 \in \dot{E}$ as $t \to (-x_0^3/3)^+$.



PROBLEM SET 2.6

1. (a) (0, 0) a source, (1, 1) and (−1, 1) saddles.

   (b) (4, 2) a source, (−2, −1) a sink.

   (c) (0, 0) a source, (0, −2), $(\pm\sqrt{3}, 1)$ saddles.

   (d) (0, 0, 0) a saddle.

   (e) See the hint concerning the origin. For $k > 1$, $\left(\pm\sqrt{k-1}, \pm\sqrt{k-1}, k-1\right)$ are sinks.

2. See Problem 1(e) regarding the nature of the equilibrium points of the Lorenz system; two new equilibrium points bifurcate from the equilibrium point $\mathbf{x} = \mathbf{0}$ at the bifurcation value $\mu = 1$ in a "pitchfork bifurcation."

3. $H^{-1}(\mathbf{x}) = (x_1, x_2 - x_1^2, x_3 - x_1^2/3)^T$ is continuous on $\mathbf{R}^3$ and if $\mathbf{y} = H(\mathbf{x})$, then $\dot{\mathbf{y}} = (\dot{x}_1, \dot{x}_2 + 2x_1\dot{x}_1, \dot{x}_3 + 2x_1\dot{x}_1/3)^T = (-x_1, -x_2 - x_1^2, x_3 + x_1^2/3)^T = (-y_1, -y_2, y_3)^T = Df(0)\mathbf{y}$.

PROBLEM SET 2.7

1. $\lambda_1 = -3$, $\lambda_2 = 7$, $\mathbf{v}_1 = (3, -2)^T$, $\mathbf{v}_2 = (1, 1)^T$, $P = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$, $P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{y} = P^{-1}\mathbf{x}$ and

$\dot{\mathbf{y}} = \text{diag}\,[-3, 7]\mathbf{y} + (-6y_1^2 + y_1y_2 + y_2^2, -9y_1^2 - 6y_1y_2 - y_2^2)^T$.

2. $\mathbf{u}^{(1)}(t, \mathbf{a}) = (e^{-t}a_1, 0)^T$, $\mathbf{u}^{(2)}(t, \mathbf{a}) = \mathbf{u}^{(3)}(t, \mathbf{a}) = (e^{-t}a_1, -e^{-2t}a_1^2/3)^T$, and $\mathbf{u}^{(j)}(t, \mathbf{a}) \to \mathbf{u}(t, \mathbf{a}) =$ $(e^{-t}a_1, -e^{-2t}a_1^2/3)^T$. Thus, $S : x_2 = -x_1^2/3$ and $U : x_1 = 0$.

3. $\boldsymbol{\phi}_t(\mathbf{c}) = (c_1e^{-t}, -c_1^2e^{-2t}/3 + (c_1^2/3 + c_2)e^t)^T$, $S : c_2 = -c_1^2/3$, for $\mathbf{x} \in S$, $\boldsymbol{\phi}_t(\mathbf{x}) =$ $(x_1e^{-t}, -x_1^2e^{-2t}/3)^T \in S$, and $U : x_1 = 0$.

4. $\mathbf{u}^{(1)}(t, \mathbf{a}) = (e^{-t}a_1, e^{-t}a_2, 0)^T$, $\mathbf{u}^{(2)}(t, \mathbf{a}) = (e^{-t}a_1, e^{-t}(a_2 + a_1^2) - e^{-2t}a_1^2, -e^{-2t}a_2^2/3)^T$, $\mathbf{u}^{(3)}(t, \mathbf{a}) =$ $\mathbf{u}^{(4)}(t, \mathbf{a}) = (e^{-t}a_1, e^{-t}(a_2 + a_1^2) - e^{-2t}a_1^2, -e^{-4t}a_1^4/5 + e^{-3t}a_1^2(a_2 + a_1^2)/2 - e^{-2t}(a_2 + a_1^2)^2/3)^T$. $S : x =$ $\psi_3(x_1, x_2)$ where $\psi_3(a_1, a_2) = u_3(0, a_1, a_2, 0) = -a_2^2/3 - a_1^2a_2/6 - a_1^4/30$; i.e., $S : x_3 =$ $-x_2^2/3 - x_1^2x_2/6 - x_1^4/30$. To find U, let $t \to -t$ to get $\dot{x}_1 = x_1$, $\dot{x}_2 = x_2 - x_1^2$ and $\dot{x}_3 =$ $-x_3 - x_2^2$. For this system $\mathbf{u}^{(1)}(t, \mathbf{a}) = \mathbf{u}^{(2)}(t, \mathbf{a}) = (e^{-t}a_1, 0, 0)^T$. Thus, $U : x_1 = 0$, $x_2 = 0$, i.e., U is the x_3-axis.

5. $x_1(t) = c_1e^{-t}$, $x_2(t) = -c_1^2e^{-2t} + (c_2 + c_1^2)e^{-t}$, $x_3(t) = -c_1^4e^{-4t}/5 + c_1^2(c_2 + c_1^2)e^{-3t}/2 -$ $(c_2 + c_1^2)^2e^{-2t}/3 + (30c_3 + c_1^4 + 5c_1^2c_2 + 10c_2^2)e^t/30$; $\lim_{t\to\infty} \boldsymbol{\phi}_t(\mathbf{c}) = \mathbf{0}$ iff $30c_3 + c_1^4 + 5c_1^2c_2 +$ $10c_2^2 = 0$; therefore, $S : x_3 = -x_2^2/3 - x_1^2x_2/6 - x_1^4/30$; and $\lim_{t\to-\infty} \boldsymbol{\phi}_t(\mathbf{c}) = \mathbf{0}$ iff $c_1 = c_2 = 0$; therefore $U : x_1 = 0$ and $x_2 = 0$.

6. Since $F \in C^1(E)$, it follows that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $\boldsymbol{\xi} \in$ $N_\delta(\mathbf{0})$, $\|DF(\boldsymbol{\xi}) - DF(\mathbf{0})\| = \|DF(\boldsymbol{\xi})\| < \varepsilon$. Thus, for all $\mathbf{x}, \mathbf{y} \in N_\delta(\mathbf{0})$, $|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})| \le \|DF(\boldsymbol{\xi})\|$ $|\mathbf{x} - \mathbf{y}| < \varepsilon\,|\mathbf{x} - \mathbf{y}|$.

7. $U_1 = \{\mathbf{x}\in S^1 \mid y > 0\}$, $h_1(x, y) = -x$, $h_1^{-1}(x) = \left(-x, \sqrt{1-x^2}\right)^T$; $U_2 = \{\mathbf{x}\in S^1 \mid y < 0\}$, $h_2(x, y) = x$, $h_2^{-1}(x) = \left(x, -\sqrt{1-x^2}\right)^T$; $U_3 = \{\mathbf{x}\in S^1 \mid x > 0\}$, $h_3(x, y) = y$, $h_3^{-1}(y) = \left(\sqrt{1-y^2}, y\right)^T$; and $U_4 = \{\mathbf{x}\in S^1 \mid x < 0\}$, $h_4(x, y) = -y$, $h_4^{-1}(y) = \left(-\sqrt{1-y^2}, y\right)^T$.

$U_1\cap U_2 = \emptyset$, $U_3\cap U_4 = \emptyset$; $h_3(U_1\cap U_3) = \{y\in \mathbf{R} \mid 0 < y < 1\}$, $h_1\circ h_3^{-1}(y) = -\sqrt{1-y^2}$ and $Dh_1\circ h_3^{-1}(y) = y/\sqrt{1-y^2} > 0$ for $y\in h_3(U_1\cap U_3)$; $h_4(U_1\cap U_4) = \{y\in \mathbf{R} \mid -1 < y < 0\}$, $h_1\circ h_4^{-1}(y) = \sqrt{1-y^2}$ and $Dh_1\circ h_4^{-1}(y) = -y/\sqrt{1-y^2} > 0$ for $y\in h_4(U_1\cap U_4)$; and it is similarly shown that $Dh_i\circ h_j^{-1}(\mathbf{x}) > 0$ for $\mathbf{x}\in h_j(U_i\cap U_j)$ for $i = 2$, $j = 3, 4$ etc.

8. $h_1\circ h_3^{-1}(z, x) = (x, \sqrt{1-x^2-z^2})$, $h_1\circ h_4^{-1}(x, z) = (x, -\sqrt{1-x^2-z^2})$, $h_1\circ h_5^{-1}(y, z) = (\sqrt{1-y^2-z^2}, y)$, $h_1\circ h_6^{-1}(z, y) = (-\sqrt{1-y^2-z^2}, y)$; $Dh_1\circ h_3^{-1}(z, x) =$

$$\begin{bmatrix} 0 & 1 \\ \dfrac{-z}{\sqrt{1-x^2-z^2}} & \dfrac{-x}{\sqrt{1-x^2-z^2}} \end{bmatrix},\quad Dh_1\circ h_4^{-1}(x, z) = \begin{bmatrix} 1 & 0 \\ \dfrac{x}{\sqrt{1-x^2-z^2}} & \dfrac{z}{\sqrt{1-x^2-z^2}} \end{bmatrix},$$

$$Dh_1\circ h_5^{-1}(y, z) = \begin{bmatrix} \dfrac{-y}{\sqrt{1-y^2-z^2}} & \dfrac{-z}{\sqrt{1-y^2-z^2}} \\ 1 & 0 \end{bmatrix},$$

$$Dh_1\circ h_6^{-1}(z, y) = \begin{bmatrix} \dfrac{z}{\sqrt{1-y^2-z^2}} & \dfrac{y}{\sqrt{1-y^2-z^2}} \\ 0 & 1 \end{bmatrix};$$

$\det Dh_1\circ h_3^{-1}(z, x) = \dfrac{z}{\sqrt{1-x^2-z^2}} > 0$ for $(z, x)\in h_3(U_1\cap U_3) = \{(z, x)\in \mathbf{R}^2 \mid x^2 + z^2 < 1, z > 0\}$, $\det Dh_1\circ h_4^{-1}(x, z) = \dfrac{z}{\sqrt{1-x^2-z^2}} > 0$ for $(x, z)\in h_4(U_1\cap U_4) = \{(x, z)\in \mathbf{R}^2 \mid x^2 + z^2 < 1, z > 0\}$, $\det Dh_1\circ h_5^{-1}(y, z) = \dfrac{z}{\sqrt{1-y^2-z^2}} > 0$ for $(y, z)\in h_5(U_1\cap U_5) = \{(y, z)\in \mathbf{R}^2 \mid y^2 + z^2 < 1, z > 0\}$, $\det Dh_1\circ h_6^{-1}(z, y) = \dfrac{z}{\sqrt{1-y^2-z^2}} > 0$ for $(z, y)\in h_6(U_1\cap U_6) = \{(z, y)\in \mathbf{R}^2 \mid y^2 + z^2 < 1, z > 0\}$, and so forth.

PROBLEM SET 2.8

1. Let $y_j(0) = y_{j0}$; then $y_1(t) = y_{10}e^{-t}$, $y_2(t) = y_{20}e^{-t} + z_0^2(e^{2t} - e^{-t})/3$, $z(t) = z_0 e^t$; $\Phi_0(\mathbf{y}, z) = (y_1, y_2)^T$, $\Phi_1(y, z) = (y_1, y_2 - e^{-2}k_0 z^2)^T$, $\Phi_2(\mathbf{y}, z) = (y_1, y_2 - e^{-2}k_0(1 + e^{-3})z^2)^T$, $\Phi_3(\mathbf{y}, z) = (y_1, y_2 - e^{-2}k_0(1 + e^{-3} + e^{-6})z^2)^T$, $\cdots$, where $k_0 = (e^3 - 1)/3e$; $\Phi_k(\mathbf{y}, z) \to (y_1, y_2 - z^2/3)^T$; $\Psi_k(\mathbf{y}, z) = z$; $H_0(\mathbf{y}, z) = (y_1, y_2 - z^2/3, z)^T$, $L^{-t}H_0T^t(\mathbf{y}, z) = (y_1, y_2 - z^2/3, z)^T$, $H(\mathbf{y}, z) = (y_1, y_2 - z^2/3, z)^T$, and $H^{-1}(\mathbf{y}, z) = (y_1, y_2 + z^2/3, z)^T$; $E^s = \{\mathbf{x} \in \mathbf{R}^3 \mid x_3 = 0\}$ and $H^{-1}(E^s) = \{\mathbf{x} \in \mathbf{R}^3 \mid x_3 = 0\}$; $E^u = \{\mathbf{x} \in \mathbf{R}^3 \mid x_1 = x_2 = 0\}$, and $H^{-1}(E^u) = \{\mathbf{x} \in \mathbf{R}^3 \mid \mathbf{x} = (0, z^2/3, z)\} = W^u(\mathbf{0})$.

2. $y(t) = y_0 e^{-t}$, $z_1(t) = z_{10}e^t$, $z_2(t) = z_{20}e^t + y_0^2(e^t - e^{-2t})/3 + y_0 z_{10}(e^t - 1)$; $\Psi_0(y, \mathbf{z}) = (z_1, z_2)^T$, $\Psi_1(y, \mathbf{z}) = (z_1, z_2 + k_0 y^2/e + k_1 y z_1/e)^T$, $\Psi_2(y, \mathbf{z}) = (z_1, z_2 + k_0 y^2(1 + e^{-3})/e + k_1 y z_1(1 + e^{-1})/e)^T$, $\Psi_3(y, \mathbf{z}) = (z_1, z_2 + k_0 y^2(1 + e^{-3} + e^{-6})/e + k_1 y z_1(1 + e^{-1} + e^{-2})/e)^T$, $\cdots$, where $k_0 = (e^3 - 1)/3e^2$ and $k_1 = e - 1$; $\Psi_k(y, \mathbf{z}) \to (z_1, z_2 + y^2/3 + yz_1)^T$; $\Phi_k(y, \mathbf{z}) = y$; $H(y, \mathbf{z}) = (y, z_1, z_2 + y^2/3 + yz_1)^T$, $H^{-1}(y, \mathbf{z}) = (y, z_1, z_2 - y^2/3 - yz_1)^T$; $E^s = \{\mathbf{x} \in \mathbf{R}^3 \mid x_2 = x_3 = 0\}$, $H^{-1}(E^s) = \{\mathbf{x} \in \mathbf{R}^3 \mid x_2 = 0, x_3 = -x_1^2/3\}$; $E^u = \{\mathbf{x} \in \mathbf{R}^3 \mid x_1 = 0\}$, $H^{-1}(E^u) = \{\mathbf{x} \in \mathbf{R}^3 \mid x_1 = 0\}$.

3. $y_1(t) = y_{10}e^{-t}$, $y_2(t) = y_{20}e^{-t} + y_{10}^2(e^{-t} - e^{-2t})$, $z(t) = z_0 e^t + y_{10}^2(e^t - e^{-2t})/3$; $\Psi_0(\mathbf{y}, z) = z$, $\Psi_1(\mathbf{y}, z) = z + k_0 y_1^2/e$, $\Psi_2(\mathbf{y}, z) = z + k_0 y_1^2(1 + e^{-3})/e$, $\Psi_3(\mathbf{y}, z) = z + k_0 y_1^2(1 + e^{-3} + e^{-6})/e$, $\cdots$, where $k_0 = (e^3 - 1)/3e^2$, and $\Psi_k(\mathbf{y}, z) \to z + y_1^2/3$; $\Phi_0(\mathbf{y}, z) = (y_1, y_2)^T$, $\Phi_1(\mathbf{y}, z) = (y_1, y_2 + k_1 e y_1^2)^T$, $\Phi_2(\mathbf{y}, z) = (y_1, y_2 + k_1 e y_1^2(1 + e^{-1}))^T$, $\Phi_3(\mathbf{y}, z) = (y_1, y_2 + k_1 e y_1^2(1 + e^{-1} + e^{-2}))^T$, $\cdots$, where $k_1 = (e - 1)/e^2$ and $\Phi_k(\mathbf{y}, z) \to (y_1, y_2 + y_1^2)^T$; $H(\mathbf{y}, z) = (y_1, y_2 + y_1^2, z + y_1^2/3)$, $H^{-1}(\mathbf{y}, z) = (y_1, y_2 - y_1^2, z - y_1^2/3)$; $E^s = \{\mathbf{x} \in \mathbf{R}^3 \mid x_3 = 0\}$, $H^{-1}(E^s) = \{\mathbf{x} \in \mathbf{R}^3 \mid x_3 = -x_1^2/3\}$; $E^u = \{\mathbf{x} \in \mathbf{R}^3 \mid x_1 = x_2 = 0\}$, $H^{-1}(E^u) = \{\mathbf{x} \in \mathbf{R}^3 \mid x_1 = x_2 = 0\}$.

5. $\Psi_m(\mathbf{z}) = (z_1, z_2 + mz_1^2)^T \to (\mathbf{z}, \infty)$. If $H(\mathbf{z})$ satisfies (9), then $H(e^2 z_1, e^4 z_2 + e^4 z_1^2) = \text{diag}[e^2, e^4]H(\mathbf{z})$; therefore, $H_2(e^2 z_1, e^4 z_2 + e^4 z_1^2) = e^4 H_2(z_1, z_2)$ and $e^4 \partial H_2/\partial z_1(z_1, z_2) = \partial H_2/\partial z_1(e^2 z_1, e^4 z_2 + e^4 z_1^2)\cdot e^2 + \partial H_2/\partial z_2(e^2 z_1, e^4 z_2 + e^4 z_1^2)\cdot 2e^4 z_1$; setting $z_1 = z_2 = 0$ implies that $\partial H_2/\partial z_1(0, 0) = 0$. A second differentiation with respect to z_1 yields $e^4 \partial^2 H_2/\partial z_1^2(0, 0) = e^2[\partial^2 H_2/\partial z_1^2\cdot e^2 + \partial^2 H_2/\partial z_1 \partial z_2\cdot 2e^4 z_1] + 2e^4 z_1 [\partial^2 H_2/\partial z_2 \partial z_1\cdot e^2 + \partial^2 H_2/\partial z_2^2\cdot 2e^4 z_1] + 2e^4 \partial H_2/\partial z_2$, the right-hand side being evaluated at $(e^2 z_1, e^4 z_2 + e^4 z_1^2)$; setting $z_1 = z_2 = 0$ then implies that $\partial^2 H_2/\partial z_1^2(0, 0) = \partial^2 H_2/\partial z_1^2(0, 0) + 2\partial H_2/\partial z_2(0, 0)$, i.e., that $\partial H_2/\partial z_2(0, 0) = 0$. Thus $J(\mathbf{z}) = \det DH(\mathbf{z}) = 0$ at $\mathbf{z} = \mathbf{0}$. Finally, if H^{-1} exists, then $H \circ H^{-1}(\mathbf{z}) = \mathbf{z}$ and then by the chain rule, if H^{-1} were differentiable at $\mathbf{z} = \mathbf{0}$ we would get $DH(H^{-1}(\mathbf{z}))\cdot DH^{-1}(\mathbf{z}) = I$ which would imply that $0 = \det DH(\mathbf{0})\cdot DH^{-1}(\mathbf{0}) = 1$, a contradiction.

PROBLEM SET 2.9

1. (a, c, d) all unstable, (b) (4, 2) is unstable and (−2, −1) is asymptotically stable, (e) $\mathbf{0}$ is asymptotically stable for $k \le 1$ and for $k > 1$, $\mathbf{0}$ is unstable and $(\pm\sqrt{k-1}, \pm\sqrt{k-1}, k-1)$ are asymptotically stable.

2. (a) (1, 0) is an unstable proper node and (−1, 0) is an unstable saddle.

   (b) (−1, −1) and (2, 2) are unstable saddles, $(\sqrt{2}, 0)$ is an asymptotically stable proper node and $(-\sqrt{2}, 0)$ is an unstable proper node.

   (c) (1, 0) is an unstable saddle and (0, 2) is an asymptotically stable node.

4. (a) $\dot{V}(\mathbf{x}) < 0$ for $\mathbf{x} \ne \mathbf{0}$ so $\mathbf{0}$ is asymptotically stable.

   (b) $\dot{V}(\mathbf{x}) > 0$ for $\mathbf{x} \ne \mathbf{0}$ so $\mathbf{0}$ is unstable.

   (c) $\dot{V}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbf{R}^2$ so $\mathbf{0}$ is a stable equilibrium point which is not asymptotically stable and solution curves lie on circles centered at the origin.

5. (a) For $V(\mathbf{x}) = x_1^2 + x_2^2$, $\dot{V}(\mathbf{x}) < 0$ for $\mathbf{x} \neq \mathbf{0}$; therefore, $\mathbf{0}$ is asymptotically stable.

(b) For $V(\mathbf{x}) = x_1^2 + x_2^2$, it follows that on any given straight line $x_2 = mx_1$ with $|m - 2| < \sqrt{3}$, $\dot{V}(\mathbf{x}) < 0$ for all sufficiently small $|\mathbf{x}| \neq 0$ and on any given straight line $x_2 = mx_1$ with $|m - 2| > \sqrt{3}$, $\dot{V}(\mathbf{x}) > 0$ for all sufficiently small $|\mathbf{x}| \neq 0$; i.e., $\mathbf{0}$ is a saddle and is unstable. This follows more easily from the Hartman-Grobman theorem since the eigenvalues of the linear part $\lambda = 1 \pm \sqrt{3}$.

(c) For $V(\mathbf{x}) = x_1^2 + 2x_2^2/3$, it follows that $\dot{V}(\mathbf{x}) < 0$ for $0 < |\mathbf{x}| < 1$; therefore, $\mathbf{0}$ is asymptotically stable. This also follows from the Hartman-Grobman theorem since the eigenvalues of the linear part $\lambda = -2 \pm i\sqrt{5}$.

(d) For $V(\mathbf{x}) = (x_1 - x_2 - 4)^4 \cdot \exp[(x_1x_2 + x_1 - x_2 + 12)/(4 + x_2 - x_1)]$, $\dot{V}(\mathbf{x}) \equiv 0$ and therefore $\mathbf{0}$ is a center. This Liapunov function can be found by making the rotation of coordinates $u = x_1 + x_2$, $v = x_1 - x_2$ to get $du/dv = (u^2/2 + v^2/2 + 4v)/(uv - 4u)$; and then letting $w = u^2$ to get $dw/dv = (w + 8v + v^2)/(v - 4)$, a linear differential equation. The solution of this linear differential equation then yields the Liapunov function $V(x_1, x_2)$. Also, note that the u, v system is symmetric with respect to the v-axis; cf. Theorem 6 in Section 2.10.

7. Let $x_1 = x$ and $\dot{x}_2 = -g(x_1)$. Then $\ddot{x} + f(x)\,\dot{x} + g(x) = 0$ is equivalent to $\ddot{x}_1 = -g(x_1) - f(x_1)\dot{x}_1 = \dot{x}_2 - F'(x_1)\dot{x}_1$ since $F'(x_1) = f(x_1)$. And this last equation is (up to an arbitrary constant) equivalent to $\dot{x}_1 = x_2 - F(x_1)$. Let $V(\mathbf{x}) = x_2^2/2 + G(x_1)$. Then $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$ if $G(x) > 0$ and $\dot{V}(\mathbf{x}) = -g(x_1)\,F(x_1) < 0$ if $g(x)\,F(x) > 0$. Thus $\mathbf{0}$ is an asymptotically stable equilibrium point.

8. $F(x) = \varepsilon(x^3 - 3x)/3$, $G(x) = x^2/2 > 0$ for $x \neq 0$, and $g(x)\,F(x) = \varepsilon x^2(x^2 - 3)/3 < 0$ for $\varepsilon > 0$ and $0 < |x| < \sqrt{3}$; therefore, for $\varepsilon > 0$ the origin is an unstable equilibrium point of the van der Pol equation.

PROBLEM SET 2.10

1. (a) $\dot r = r$, $\dot\theta = 1$: the origin is an unstable focus.

   (b) $\dot r = ry^2$, $\dot\theta = 1$: the origin is an unstable focus.

   (c) $\dot r = (x^6 + y^6)/r > 0$ and $\dot\theta = 1 + xy(y^4 - x^4)/r^2 > 0$ for sufficiently small $r > 0$: the origin is an unstable focus.

3. Let $\mathbf{F}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{0})\mathbf{x}$. Then according to the definition of differentiability, Definition 1 in Section 2.1, $|\mathbf{F}(\mathbf{x})| / |\mathbf{x}| \to 0$ as $|\mathbf{x}| \to 0$, i.e., as $\mathbf{x} \to \mathbf{0}$.

4. (a) (0, 0) is an unstable proper node, (1, 1) and (−1, 1) are topological saddles.

   (b) (4, 2) is an unstable node and (−2, −1) is a stable focus.

   (c) (0, 0) is an unstable proper node and (0, −2), $(\pm\sqrt{3}, 1)$ are topological saddles.

   (d) (0, 1) is a center since the system is symmetric with respect to the y-axis and (0, −1) is a topological saddle.

   (e) (0, ±1) are centers since the system is Hamiltonian and also since it is symmetric with respect to the y-axis and (±1, 0) are topological saddles.

   (f) (1, 0) is an unstable node and (−1, 0) is a topological saddle.

PROBLEM SET 2.11

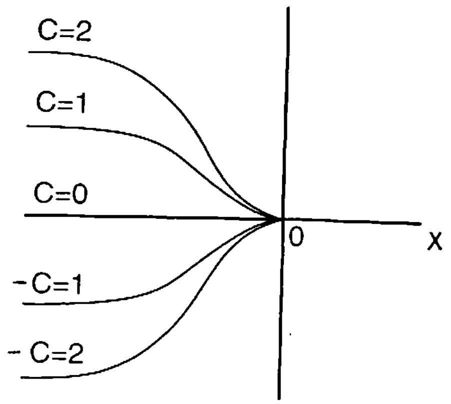
1. In Theorem 2, n = m = 1 is an odd integer, $b_1 = 4 \neq 0$ and $\lambda = 8 > 0$; therefore the system has a critical point with an elliptic domain at the origin. For $V(\mathbf{x}) = y - x^2/(2 \pm \sqrt{2})$ we have $\dot V(\mathbf{x}) \equiv 0$ on $y = x^2/(2 \pm \sqrt{2})$; thus $y = x^2/(2 \pm \sqrt{2})$ are invariant curves of the system. This system is best understood by drawing its global phase portrait; cf. Section 3.10.

2. (a, b, e, f) $\mathbf{0}$ is a saddle-node. (c) $\mathbf{0}$ is a node (and it is unstable). (d) $\mathbf{0}$ is a topological saddle.

3. (a, b) **0** os a cusp. (c) **0** is a saddle-node. (d) **0** is a focus or center according to Theorem 2 and using $V(\mathbf{x}) = x^4 + 2y^2$ with $\dot{V}(\mathbf{x}) = -4x^2y^2$, it is a stable focus. (e) **0** is a topological saddle. (f) **0** is a focus or center according to Theorem 2; use the coordinate ransformation $\xi = x$, $\eta = x + y$ to put the system into the normal form (3). Also, it can be shown that **0** is a stable focus.

PROBLEM SET 2.12

1. Substituting $h(x) = a_2 x^2 + a_3 x^3 + \cdots$ into (5) yields $a_2 = 0$ and $na_n + a_{n+1} = 0$ for integer $n \geq 2$; and this implies that $a_1 = a_2 = \cdots = 0$, i.e., that $h(x) \equiv 0$. For the function $h(x, c)$ given in this problem, we have $h'(x, c) = 0$ for $x \geq 0$ and $h'(x, c) = -ce^{1/x}/x^2$ for $x < 0$. Substitution into equation (5) yields $0 = 0$ for $x \geq 0$ and $-ce^{1/x}/x^2[x^2] - (-ce^{1/x}) = 0$ for $x < 0$; i.e., $h(x, c)$ satisfies equation (5) for all $x \in \mathbf{R}$. Also, since $h(x, c)$ is (real) analytic at each point $x \neq 0$ with $h^{(n)}(x, c) \to 0$ as $x \to 0$ and since $h^{(n)}(0, c) = 0$ for all $n = 1, 2, \cdots$, it follows that $h(x, c) \in C^\infty(\mathbf{R})$.



2. Diagonalization yields a system of the form $\dot{x} = \alpha(x + y)^2 - y(x + y)$, $\dot{y} = -y - \alpha(x + y)^2 + y(x + y)$; then from (5), $h(x) = -\alpha x^2 + \alpha x^3 + \cdots$ and on $W^c(\mathbf{0})$, $\dot{x} = \alpha x^2 + 0(x^3)$; so for $\alpha \neq 0$, **0** is a saddle-node. For $\alpha = 0$, $h(x) \equiv 0$ and the x-axis is a line of critical points.

3. The linear part of the system is already in diagonal form and from (5), $h(x) = -x^2 - 2x^4 + \cdots$; on $W^c(\mathbf{0})$, $\dot{x} = -x^3 + \cdots$ and the origin is a stable node.

4. From (5) we have for $h(x) = a_2x^2 + a_3x^3 + \cdots$ that $(2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots)(-x^3) + (a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots) - x^2 = 0$ identically in x for $|x| < \delta$. Therefore setting the coefficients of like powers of x equal to zero yields $a_2 = 1$, $a_3 = 0$, $2a_2 = a_4$, $a_5 = 0$, $4a_4 = a_6$, $a_7 = 0$, $\cdots$, i.e., $a_2 = 1$, $a_4 = 2a_2 = 2$, $a_6 = 4a_4 = 2.4$, $\cdots$, $a_{2n} = 2^n n!$ and $a_{2n+1} = 0$. Thus,

$h(x) = \sum_{n=0}^{\infty} 2^n n!\, x^{2n+2}$ diverges except at $x = 0$ and this polynomial system has no analytic center manifold. However, since $\dot{x} = -x^3 < 0$ for $x > 0$ and $\dot{x} = -x^3 > 0$ for $x < 0$, any trajectory $\gamma^{\pm}(t)$ with $\gamma^{\pm}(0) = (x^{\pm}(0)), y^{\pm}(0))$ and $x^{+}(0) > 0$ or $x^{-}(0) < 0$ can be represented by a function $y = f^{\pm}(x)$ which is analytic for $x > 0$ or $x < 0$ respectively. And since $W^c(\mathbf{0})$ is invariant under the flow, it follows from Theorem 1 that given $f^{+}(x)$, there exists an $f^{-}(x)$ such that the function $h(x) = \{f^{+}(x) \text{ for } x > 0, 0 \text{ for } x = 0, f^{-}(x) \text{ for } x < 0\}$ represents a C^∞ center manifold, $W^c(\mathbf{0})$, and $\dot{x} = -x^3$ on $W^c(\mathbf{0})$; thus, the origin is a stable node.

5. (a) From (5), $h(\mathbf{x}) = -x_1^2 - x_2^2 + \cdots$; on $W^c(\mathbf{0})$, $\dot{x}_1 = -x_2 + 0(|\mathbf{x}|^3)$, $\dot{x}_2 = x_1 + 0(|\mathbf{x}|^3)$ and the origin is topologically a stable focus on $W^c(\mathbf{0})$ which follows using the Liapunov function $V(\mathbf{x}) = (x_1^2 + x_2^2)/2$ or by showing that $\dot{r} = -r^3 + 0(r^4)$ and $\dot{\theta} = 1 + 0(r)$ for the system on $W^c(\mathbf{0})$; hence, $\mathbf{0}$ is a symptotically stable critical point.

   (b) There is a saddle-node at the origin on $W^c(\mathbf{0})$.

   (c) There is a critical point with two hyperbolic sectors at the origin on $W^c(\mathbf{0})$.

6. Let $h(x) = a_2 x^2 + a_3 x^3 + \cdots$; then from (5), $h(x) = dx^2 + (ed - 2ad)x^3 + \cdots$ and on $W^c(\mathbf{0})$, $\dot{x} = ax^2 + bdx^3 + 0(x^4)$. Thus, for $a \neq 0$, the origin is a saddle-node; for $a = 0$ and $bd > 0$, the origin is a saddle; for $a = 0$ and $bd < 0$, the origin is a stable node; for $a = b = 0$ and $cd \neq 0$, $\dot{x} = cd^2 x^4 + 0(x^5)$ on $W^c(\mathbf{0})$ and the origin is a saddle-node. If $a = d = 0$, the x-axis consists of critical points.

7. $h(\mathbf{x}) = x_1^2 + 0(|\mathbf{x}|^3)$; on $W^c(\mathbf{0})$, $\dot{x}_1 = -x_1^3 - x_2^3 + 0(|\mathbf{x}|^4)$; $\dot{x}_2 = x_1^3 - x_2^3 + 0(|\mathbf{x}|^4)$ and the origin is topologically a stable focus on $W^c(\mathbf{0})$ which follows using the Liapunov function $V(\mathbf{x}) = (x_1^4 + x_2^4)/4$; hence $\mathbf{0}$ is an asymptotically stable critical point.

PROBLEM SET 2.13

1. $L_J[\mathbf{h}_2(\mathbf{x})] = (b_{20}x^2 + (b_{11} - 2a_{20})xy + (b_{02} - a_{11})y^2, -2b_{20}xy - b_{11}y^2)^T$ and for $a_{02} = a_{11} = 0$, $a_{20} = (b + f)/2$, $b_{02} = -c$, $b_{11} = f$ and $b_{20} = -a$, $L_J[\mathbf{h}_2(\mathbf{x})] + \mathbf{F}_2(\mathbf{x}) = (0, dx^2 + (e + 2a)xy)^T$.

2. Since $L_J[\mathbf{h}_3(\mathbf{x})] = b_{30}(x^3, -3x^2y)^T + b_{21}(x^2y, -2xy^2)^T + b_{12}(xy^2, -y^3)^T + (b_{03} - a_{12})(y^3, 0)^T - 3a_{30}(x^2y, 0)^T - 2a_{21}(xy^2, 0)^T$, the result for $L_J(H_3)$ follows; and then it is clear that $H_3 = L_J(H_3) \oplus G_3$.

3. As in the paragraph preceding Remark 1, for $\mathbf{F}_2 = \tilde{\mathbf{F}}_2 = 0$, the system $\dot{\mathbf{x}} = J\mathbf{x} + F_3(\mathbf{x}) + 0(|\mathbf{x}|^4)$ can be reduced, by letting $\mathbf{x} = \mathbf{y} + \mathbf{h}_3(\mathbf{y})$, to a system of the form $\dot{\mathbf{y}} = J\mathbf{y} + \tilde{\mathbf{F}}_3(\mathbf{y}) + 0(|\mathbf{x}|^4)$ with $\tilde{\mathbf{F}}_3 \in G_3$, i.e., to a system of the form $\dot{x} = y + 0(|\mathbf{x}|^4)$, $\dot{y} = ax^3 + bx^2y + 0(|\mathbf{x}|^4)$ for $a, b \in \mathbf{R}$. And letting $y + 0(|\mathbf{x}|^4) \to y$, we get a system of the form (3) in Section 2.11; according to Theorem 2 in 2.11, for $a > 0$ there is a topological saddle at the origin and for $a < 0$ there is a focus or a center at the origin.

4. Similar to Problem 3, we get a system of the form (3) in 2.11: $\dot{x} = y$, $\dot{y} = ax^4 + bx^3y + 0(|\mathbf{x}|^5)$ which, for $a \neq 0$, has a cusp at the origin.

5. For $x_1 = y_1$ and $x_2 = y_2 - y_1^2$, the given system reduces to $\dot{x} = y - x^3 + xy^2 - y^3 + 0(|\mathbf{x}|^4)$, $\dot{y} = x^2 + 3x^3 + x^2y + 0(|\mathbf{x}|^4)$ and then for $\mathbf{x} = (y_1, y_2 + y_1^3 - y_1y_2^2 + y_2^3)^T$ or $\mathbf{y} = (x_1, x_2 - x_1^3 + x_1x_2^2 - x_2^3)^T$, this system reduces to $\dot{y}_1 = y_2 + 0(|\mathbf{x}|^4)$; $\dot{y}_2 = y_1^2 + 3y_1^3 - 2y_1^2y_2 + 0(|\mathbf{x}|^4)$.

PROBLEM SET 2.14

1. (a) $H(x, y) = a_{11}xy + a_{12}y^2/2 - a_{21}x^2/2 + Ax^2y - Bxy^2 + Cy^3/3 - Dx^3/3$. (b) If $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is Hamiltonian, then $\mathbf{f} = (H_y, -H_x)$ for $\mathbf{x} \in E$ and therefore $\nabla \cdot \mathbf{f} = \partial H_y/\partial x - \partial H_x/\partial y = 0$ for $\mathbf{x} \in E$. On the other hand, if $\nabla \cdot \mathbf{f} = 0$, i.e., if $\partial f_1/\partial x = -\partial f_2/\partial y$ in a simply connected region E, then the first-order differential equation $-f_2 dx + f_1 dy = 0$ is exact. (See, for example, Theorem 2.8.1 in W.E. Boyce and R.C. Di Pima, "Elementary Differential Equations and Boundary Value Problems," J. Wiley, NY, 1997.) Thus, there exists a function $H \in C^2(E)$ such that $dH = H_x dx + H_y dy = -f_2 dx + f_1 dy$ and therefore the system $\dot{x} = f_1 = H_y$, $\dot{y} = f_2 = -H_x$ is Hamiltonian on E.

2. $H(x, y) = T(y) + U(x) = y^2/2 + x^2/2 - x^3/3$; U(x) has a strict local minimum at $x = 0$ and a strict local maximum at $x = 1$; and therefore the Hamiltonian system has a center at (0, 0) and a saddle at (1, 0).

3. $H(x, y) = y^2/2 + x^2/2 - x^4/4$; there is a center at (0, 0) and saddles at (±1, 0).

5. (a) The Hamiltonian system has a center and the gradient system has a stable node at (0, 0).

   (c) The Hamiltonian and gradient systems have saddles at $(n\pi, 0)$ for $n \in \mathbf{Z}$.

   (e) The Hamiltonian system has a center and the gradient system has a stable node at (–4/3, –2/3).

6. (a) The surfaces V(x, y, z) = constant are paraboloids with their vertices on the z-axis and trajectories, other than the z-axis, approach the positive z-axis asymptotically as $t \to \infty$.

   (b) The surfaces V(x, y, z) = constant are concentric ellipsoids and the origin is a stable, three-dimensional node.

   (c) Each of the surfaces V(x, y, z) = constant has a strict local maximum at the origin, a strict local minimum at (2/3, 4/3, 0) and saddles at (2/3, 0, 0) and (0, 4/3, 0); the gradient system has a source at the origin, a sink at (2/3, 4/3, 0) and saddles at (2/3, 0, 0) and (0, 4/3, 0).

7. Since $\mathbf{x}_0$ is a strict local minimum of $V(\mathbf{x})$, there is a $\delta > 0$ such that $V(\mathbf{x}) - V(\mathbf{x}_0) > 0$ for $0 < |\mathbf{x}| < \delta$ and $d/dt[V(\mathbf{x}) - V(\mathbf{x}_0)] = [\partial V/\partial \mathbf{x}]\cdot\dot{\mathbf{x}} = -[(\partial V/\partial x_1)^2 + \cdots + (\partial V/\partial x_n)^2] < 0$ for $0 < |\mathbf{x}| < \delta$.

9. First of all $(x_0, 0)$ is a critical point of the Newtonian system (3) iff $U'(x_0) = 0$. Since $\det Df(x_0, 0) = U''(x_0)$ and trace $Df(x_0, 0) = 0$, it follows that $(x_0, 0)$ is a saddle of the Newtonian system (3) if $U''(x_0) < 0$, i.e., if x_0 is a strict local maximum of $U(x)$; and since (3) is symmetric with respect to the x-axis, it follows that $(x_0, 0)$ is a center of the Newtonian system (3) if $U''(x_0) > 0$, i.e., if x_0 is a strict local minimum of $U(x)$; finally, if x_0 is a horizontal inflection point of $U(x)$, then $U'(x_0) = 0$ and the first nonvanishing derivative of $U(x)$ at x_0 is odd; therefore, it follows from Theorem 3 in Section 2.11 that $(x_0, 0)$ is a cusp for the Newtonian system (3).

11. Let $x_1 = x$, $x_2 = y$, $y_1 = \dot{x}$ and $y_2 = \dot{y}$. The two-body problem is a Hamiltonian system with $H(x_1, x_2, y_1, y_2) = (y_1^2 + y_2^2)/2 - (x_1^2 + x_2^2)^{-1/2}$. The gradient system orthogonal to this system is $\dot{x}_1 = -x_1/(x_1^2 + x_2^2)^{3/2}$, $\dot{x}_2 = -x_2/(x_1^2 + x_2^2)^{3/2}$, $\dot{y}_1 = -y_1$, $\dot{y}_2 = -y_2$.

12. By Problem 1(b), if $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is Hamiltonian, then $\nabla\cdot\mathbf{f} = 0$ in E (even if E is not simply connected) and then by Problem 6 in Section 2.3, the flow defined by this system is area preserving.

3. NONLINEAR SYSTEMS: GLOBAL THEORY

PROBLEM SET 3.1

1. $$\phi(t, x) = e^{At}x = \begin{bmatrix} e^{-t} & \left(e^{-t} - e^{2t}\right) \\ 0 & e^{2t} \end{bmatrix} \mathbf{x}.$$

2. The differential equation $\dot{x} = x^2/(1 + x^2)$ is separable; its solution is $x(t) = \left(t + c \pm \sqrt{(t+c)^2 + 4}\right)/2$; for $x_0 \neq 0$, $x(0) = x_0$ if $c = x_0 - 1/x_0$ and the $\pm$ sign is chosen as $x_0/|x_0|$ and this yields the result in Example 1; for $x_0 = 0$ the solution is $x(t) \equiv 0$.

3. If $f(x) \neq 0$ at $x \in E$, then $D|f(x)| = |f(x)|\, f'(x)/f(x)$; and this then yields $DF(x) = f'(x)/(1 + |f(x)|)^2$; if $f(x_0) = 0$ at $x_0 \in E$, then $DF(x_0) = \lim_{h\to 0} [F(x_0 + h) - F(x_0)]/h = \lim_{h\to 0} f(x_0 + h)/(1 + |f(x_0 + h)|)/h = f'(x_0)$ and then $\lim_{x\to x_0} DF(x) = \lim_{x\to x_0} f'(x)/(1 + |f(x)|)^2 = f'(x_0)$ since $f' \in C(E)$ and since $f(x_0) = 0$; hence $F \in C^1(E)$.

4. $f'(x) = -2x/(1 + x^2)^2$ and $f'(x)$ assumes its maximum/minimum at $x = \mp 1/\sqrt{3}$; thus $|f'(x)| \leq |f'(\pm 1/\sqrt{3})| = 3\sqrt{3}$; then by the mean value theorem $|f(x) - f(y)| \leq 3\sqrt{3}\,|x - y|$ for $x, y \in \mathbf{R}$. The differential equation $\dot{x} = 1/(1 + x^2)$ is separable and its solution satisfying $x(0) = x_0$ is given by the solution of the cubic $x^3 + 3x - (3t + k_0) = 0$ with $k_0 = x_0^3 + 3x_0$; the solution of this cubic is $x(t) = \left\{\left[(3t + k_0) + \sqrt{(3t + k_0)^2 + 4}\right]^{1/3} + \left[(3t + k_0) - \sqrt{(3t + k_0)^2 + 4}\right]^{1/3}\right\}/2^{1/3}$ and $x(t) \to \pm\infty$ as $t \to \pm\infty$.

6. (a) If $\mathbf{x}_0$ is an equilibrium point of (1) then $\phi_t(\mathbf{x}_0) = \mathbf{x}_0$ for all $t \in \mathbf{R}$; and since $\tau(\mathbf{x}_0, t)$ maps $\mathbf{R}$ onto $\mathbf{R}$, it follows that $\psi_\tau(H(\mathbf{x}_0)) = H(\phi_t(\mathbf{x}_0)) = H(\mathbf{x}_0)$ for all $\tau \in \mathbf{R}$; i.e., $H(\mathbf{x}_0)$ is an equilibrium point of (2). Alternatively, one may follow the hint given in Problem 6.

(b) If $\phi_t(x_0)$ is a periodic solution of (1) with period t_0, then $\phi_{t_0}(x_0) = x_0$ and therefore if $\tau_0 = \tau(x_0, t_0)$, it follows that $\psi_{\tau_0}(H(x_0)) = H(\phi_{t_0}(x_0)) = H(x_0)$, i.e., $\psi_\tau(H(x_0))$ is a periodic solution of (2) of period τ_0.

7. (Cf. [Wi], p. 25–26.) Differentiating (*) wih respect to t yields $DH(\phi_t(x))\partial\phi_t(x)/\partial t = \partial\tau(x, t)/\partial t\cdot\partial\psi_\tau(H(x))/\partial\tau$ which at $t = 0$ yields $DH(x)f(x) = \partial\tau(x, 0)/\partial t\cdot g(H(x))$. Then differentiating this last equation with respect to x yields $D^2H(x)f(x) + DH(x)Df(x) = \partial\tau(x, 0)/\partial t\cdot Dg(H(x))DH(x) + \partial^2\tau(x, 0)/\partial x\partial t\cdot g(H(x))$. And then setting $x = x_0$, this yields $ADf(x_0)A^{-1} = \partial\tau(x_0, 0)/\partial t\cdot Dg(H(x_0))$. Thus, the eigenvalues of $Df(x_0)$ and the eigenvalues of $Dg(H(x_0))$ are related by the positive constant $k_0 = \partial\tau(x_0, 0)/\partial t$.

9. For $F(x, y) = (y, \mu x + y - y^3)$ and $\mu \neq 0$, $F^{-1}(x, y) = (y - x + x^3, \mu x)/\mu$;
$DF(x, y) = \begin{bmatrix} 0 & 1 \\ \mu & 1-3y^2 \end{bmatrix}$ and $DF^{-1}(x, y) = \begin{bmatrix} (-1+3x^2)/\mu & 1/\mu \\ 1 & 0 \end{bmatrix}$ are continuous; and an easy computation yields $F(\sqrt{\mu}, \sqrt{\mu}) = (\sqrt{\mu}, \sqrt{\mu})$.

PROBLEM SET 3.2

1. There is a saddle at (0, 0) and stable nodes at (±1, 0). [–1, 1] is an attracting set, but it is not an attractor since it does not contain a dense orbit. (0, 1] is not an attractor since it is not closed. [1, ∞) is an attractor. (0, ∞), [0, ∞) and (–1, ∞) are not attracting sets. [–1, ∞) and (–∞, ∞) are attracting sets.

2. (a) By the theorem of Hurwitz given in this problem, for any irrational number α and any integer $N > 0$, there are positive integers m, n such that $n > N$ and $|\alpha n - m| < 1/n$. Furthermore, for any $\varepsilon > 0$, if we choose $N \geq 2\pi/\varepsilon$, then $|2\pi\alpha n - 2\pi m| < 2\pi/n \leq \varepsilon$ and then for $a = \exp[2\pi i\alpha]$, $|a^n - 1| < \varepsilon$. Let $\theta = 2\pi n\alpha - 2\pi m$; then $0 < |\theta| < \varepsilon$ and there exists an integer K such that $K|\theta| < 2\pi < (K + 1)|\theta|$. Thus, for any point a_0 on the unit circle C, there is an integer $j \in \{1, \cdots, K\}$ such that $|a^j - a_0| < \varepsilon$; therefore $\{a^k \mid k = 1, 2, \cdots\}$ is dense in C.

(b) The flow $\phi_t(w, z) = (e^{2\pi i t}w, e^{2\pi\alpha i t}z)$; it follows that $\phi_n(w, z) = (e^{2\pi i n}w, e^{2\pi\alpha n i}z) = (w, a^n z)$.

(c) Let $x_0 = (w_0, z_0) \in T^2$. Given any point $(w_1, z_1) \in T^2$, let $t_0 = \arg w_1 - \arg w_0$. Then $e^{2\pi i t_0}w_0 = w_1$ and for $\tilde{z}_0 = e^{2\pi\alpha i t_0}z_0$, $\phi_{t_0}(w_0, z_0) = (w_1, \tilde{z}_0)$ since $\phi_t(w_0, z_0) = (e^{2\pi i t}w_0, e^{2\pi\alpha i t}z_0)$. Then for any $z_1 \in C$ and $\varepsilon_n = 1/n$, there is a positive integer $k_n > n$ such that $|\tilde{z}_0 \exp[2\pi\alpha i k_n] - z_1| < 1/n$: this follows from part (a) with $a_0 = z_1/\tilde{z}_0$ and $\varepsilon = 1/n$. Thus, for any point $(w_1, z_1) \in T^2$, if we let $t_n = t_0 + k_n$, then $t_n \to \infty$ and $\phi_{t_n}(w_0, z_0) = \phi_{k_n} \circ \phi_{t_0}(w_0, z_0) = \phi_{k_n}(w_1, \tilde{z}_0) \to (w_1, z_1)$ as $n \to \infty$; therefore, $(w_1, z_1) \in \omega(\Gamma_{x_0})$; i.e., $\omega(\Gamma_{x_0}) = T^2$. Similarly, it is shown that $\alpha(\Gamma_{x_0}) = T^2$.

(d) Any trajectory of this system is a solution of the Hamiltonian system with two degrees of freedom $\dot{x} = -2\pi\alpha y$, $\dot{y} = 2\pi\alpha x$; $\dot{u} = -2\pi v$, $\dot{v} = 2\pi u$; with $H(\mathbf{x}) = -\pi[\alpha(x^2 + y^2) + (u^2 + v^2)]$. Thus, trajectories lie on the ellipsoidal surfaces $E_k = \{\mathbf{x} \in \mathbf{R}^4 \mid \alpha(x^2 + y^2) + (u^2 + v^2) = k^2\}$. For a given $k \in \mathbf{R}$ and $\mathbf{x}_0 \in E_k$, it follows from part (c) that $\omega(\Gamma_{x_0})$ is the torus $T^2_{h,k} = \{\mathbf{x} \in \mathbf{R}^4 \mid u^2 + v^2 = h^2, x^2 + y^2 = (k^2 - h^2)/\alpha\} = C_h \times C_{h'}$ with $h' = \sqrt{(k^2 - h^2)/\alpha}$ and, as in Section 3.6, for a given $k \in \mathbf{R}$, we can project from the north pole of the surface E_k to obtain the projection of the tori $T^2_{h,k}$ onto $\mathbf{R}^3$; cf. Figure 5 in Section 3.6.

3. Reflexive: $\Gamma_1 \sim \Gamma_1$ since $\phi_t(\mathbf{x}_1) = \phi_{t+t_0}(\mathbf{x}_1)$ for $t_0 = 0$. Symmetric: If $\Gamma_1 \sim \Gamma_2$ then $\phi_t(\mathbf{x}_2) = \phi_{t+t_0}(\mathbf{x}_1)$ which is equivalent to $\phi_{t-t_0}(\mathbf{x}_1) = \phi_t(\mathbf{x}_1)$, i.e., $\Gamma_2 \sim \Gamma_1$. Transitive: If $\Gamma_1 \sim \Gamma_2$ and $\Gamma_2 \sim \Gamma_3$ then $\phi_t(\mathbf{x}_2) = \phi_{t+t_0}(\mathbf{x}_1)$ and $\phi_t(\mathbf{x}_3) = \phi_{t+t_1}(\mathbf{x}_2) = \phi_{t+t_0+t_1}(\mathbf{x}_1)$, thus $\Gamma_1 \sim \Gamma_3$. This equivalence relation partitions the set of solution curves of (1) into equivalence classes called trajectories.

4. $\omega(\Gamma)$ cannot consist of one limit orbit and two equilibrium points; in case (d) there are two different topological types given by the top two figures in Figure 4 in Section 3.3.

6. (a) Replacing x by $-x$ and y by $-y$ does not change the system.

(b) For $x = y = 0$, $\dot{x} = 0$, and $\dot{y} = 0$, so the z-axis is invariant and consists of three trajectories: the origin together with the positive and negative z-axes.

(c) Substituting the coordinates for the equilibrium points into the right-hand side of the system gives zero; for $\sigma > 0$ and $\rho > 1$, the linear part at the origin has two negative eigenvalues and one positive eigenvalue.

(d) $\dot{V}(\mathbf{x}) = -2\sigma[(\rho x - y)^2 + \beta z^2] < 0$ except on the line $z = 0$, $y = \rho x$; thus, for $0 < \rho < 1$, $\mathbf{0}$ is globally asymptotically stable.

PROBLEM SET 3.3

1. (a) $\dot{r} = r(1 - r^2)\sin\left[1/\sqrt{|1-r^2|}\right]$, $\dot{\theta} = 1$ and $\dot{r} = 0$ if $r = \sqrt{1 \pm (1/n^2\pi^2)}$. This defines a sequence of limit cycles $\Gamma_n^{\pm}$ which approach the cycle Γ on $r = 1$; the limit cycles $\Gamma_n^{\pm}$ are stable for n odd and unstable for n even.

(b) Similarly, $\dot{\theta} = 1$, $\dot{r} = r(1 - r^2)\sin[1/(1 - r^2)] = 0$ if $r = \sqrt{1-(1/n\pi)}$, n a nonzero integer; Γ_n is stable for n odd and positive or n even and negative and Γ_n is unstable for n even and positive or odd and negative.

2. $\dot{\theta} = 1$ and $\dot{r} = r(1 - r^2)^2 = 0$ if $r = 1$ and $\dot{r} > 0$ for $r \neq 0$ or 1.

3. From the example in Section 1.5 we have a one-parameter family of cycles lying on the ellipses $x(t) = \alpha\cos 2t$, $y(t) = (\alpha/2)\sin 2t$ with parameter $\alpha \in (0, \infty)$ and period $T_\alpha = \pi$.

(b) $\dot{r} \equiv 0$ and $\dot{\theta} = r > 0$ for $r > 0$; by substitution into the system of differential equations, $x(t) = \alpha\cos\alpha t$, $y(t) = \alpha\sin\alpha t$ is a periodic solution with period $T_\alpha = 2\pi/\alpha$ for $\alpha \in (0, \infty)$.

4–6. Use the result of Problem 1 in Section 2.14 to show that the system is Hamiltonian and then use Theorem 2 in Section 2.14 to determine which critical points are saddles and which are centers.

7. (a) $\dot{\theta} = 1$, $\dot{r} = r(1 - r^2)(4 - r^2)$ has two limit cycles $\Gamma_1 : \gamma_1(t) = (\cos t, \sin t)^T$ and $\Gamma_2 : \gamma_2(t) = (2\cos t, 2\sin t)^T$; Γ_1 is stable and Γ_2 is unstable; $\mathbf{0}$, Γ_1, Γ_2 are the only limit sets of this system.

(b) $\dot{\theta} = 1$, $\dot{r} = r(1 - r^2 - z_0^2)(4 - r^2 - z_0^2)$ for $z = z_0$; the spheres $S_1 : r^2 + z^2 = 1$ and $S_2 : r^2 + z^2 = 4$ are invariant; there is no attracting set; cf. Example 2 in Section 3.2.

8. $\dot{\theta} = 1$, $\dot{r} = r(1 - r^2)(4 - r^2)$, $z(t) = z_0 e^t$; Γ_1 and Γ_2 are unstable, $W^s(\Gamma_1) = \{\mathbf{x} \in \mathbf{R}^3 \mid z = 0, 0 < r < 2\}$, $W^u(\Gamma_1) = \{\mathbf{x} \in \mathbf{R}^3 \mid r = 1\}$, $W^s(\Gamma_2) = \Gamma_2$, $W^u(\Gamma_2) = \{\mathbf{x} \in \mathbf{R}^3 \mid 1 < r < \infty\}$. The unit cylinder is the only attracting set for this system.

9. Γ_2 is a stable periodic orbit $W^s(\Gamma_2) = \{\mathbf{x} \in \mathbf{R}^3 \mid 1 < r < \infty\}$; the origin, Γ_2, the z-axis and the cylinder $r = 2$ are attracting sets for this system; and the origin and Γ_2 are the only attractors for this system.

PROBLEM SET 3.4

1. Substitution into the system of differential equations shows that $\gamma(t)$ is a periodic solution. Since $\nabla \cdot \mathbf{f}(\gamma(t)) = -2$ (since $1 - x^2/4 - y^2 = 0$ on $\gamma(t)$), it follows from the corollary to Theorem 2 that Γ is a stable limit cycle.

2. Substitution shows that $\gamma(t)$ is a periodic solution. In cylindrical coordinates $\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1$ and $\dot{z} = z$, which has the solution $\phi_t(r_0, \theta_0, z_0) = ([1 + (1/r_0^2 - 1)e^{-2t}]^{-1/2}, t + \theta_0, z_0 e^t)^T$; thus $P(r_0, z_0) = ([1 + (1/r_0^2 - 1)e^{-4\pi}]^{-1/2}, z_0 e^{2\pi})^T$, $DP(r_0, z_0) = \text{diag}\left[e^{-4\pi} r_0^{-3} \cdot [1 + (1/r_0^2 - 1)e^{-4\pi}]^{-3/2}, e^{2\pi}\right]$ and $DP(1, 0) = \text{diag}\,[e^{-4\pi}, e^{2\pi}] = e^{2\pi B}$ where $B = \text{diag}\,[-2, 1]$.

3. (a) For $\mathbf{x}_0 = (x_0, 0)$, $\phi_t(\mathbf{x}_0) = e^{at} R_{bt}\, \mathbf{x}_0 = e^{at}(x_0 \cos bt, x_0 \sin bt)^T$; at $t = 2\pi/|b|$, we get $P(x_0) = x_0 \exp[2\pi a/|b|]$; for $d(x) = P(x) - x = x \exp[2\pi a/|b|] - x$, $d'(0) = d'(x) = \exp[2\pi a/|b|] - 1$ and clearly $d(-x) = -d(x)$.

(b) $P(s) = [1 + (1/s^2 - 1)e^{-4\pi}]^{-1/2}$ for $s \neq 0$ and $P(0) = 0$; and this is equivalent to $P(s) = s[s^2 + (1 - s^2)e^{-4\pi}]^{-1/2}$ which is (real) analytic for all $s \in \mathbf{R}$ since $s^2 + (1 - s^2)e^{-4\pi} = e^{-4\pi} + (1 - e^{-4\pi})s^2 > 0$ for all $s \in \mathbf{R}$; since $P'(s) = e^{-4\pi}[s^2(1 - e^{-4\pi}) + e^{-4\pi}]^{-3/2}$ for all $s \in \mathbf{R}$, $P'(0) = e^{2\pi}$ and $d'(0) = e^{2\pi} - 1 > 0$; thus, the origin is a simple focus which is unstable.

4. $\dot\theta = 1$, $\dot r = r(1 - r^2)^2$ and $\gamma(t) = (\cos t, \sin t)^T$ is a semi-stable limit cycle of this system; since $\nabla\cdot\mathbf{f}(\gamma(t)) \equiv 0$, it follows from Theorem 2 that $d(0) = d'(0) = 0$ and hence $k \geq 2$ in Definition 2, i.e., Γ is a multiple limit cycle.

5. If $a = 0$, $b \neq 0$, $a_{20} + a_{02} = b_{20} + b_{02} = 0$, then according to equation (3), $\sigma = d'''(0) = 0$ and therefore the first non-vanishing derivative $d^{(k)}(0) \neq 0$ has $k = 2m + 1 \geq 5$, i.e., the origin is either a center or a focus of multiplicity $m \geq 2$.

PROBLEM SET 3.5

1. Direct substitution shows that $\gamma(t)$ is a periodic orbit of the system. The linearization about $\gamma(t)$ has $A = Df(\gamma(t)) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $\Phi(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$ as its fundamental matrix satisfying $\Phi(0) = I$. It follows that $\Phi(t) = Q(t)e^{Bt}$ with $Q(t)$ given in Example 1 and $B = \text{diag}\,[0, 0, -1]$; therefore, the characteristic exponents of $\gamma(t)$ are $\lambda_1 = 0$ and $\lambda_2 = -1$ and the characteristic multipliers are 1 and $e^{-2\pi}$; dim $S(\Gamma) = 2$, dim $C(\Gamma) = 2$ and dim $U(\Gamma) = 1$.