## I. *On the analytical representation of a uniform branch of a monogenic function.*

By G. Mittag-Leffler.

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Let $a$ denote a point in the plane of the complex variable $x$, and associate with $a$ an unlimited array of quantities

$$F(a),\ F^{(1)}(a),\ F^{(2)}(a), \dots,\ F^{(\mu)}(a), \dots \qquad (1),$$

where each quantity is completely determinate when the position which it occupies in the array is known.

Suppose that, as is possible in an infinite number of ways, these quantities $F$ are chosen so that Cauchy's condition*, that the series

$$P(x|a) = \sum_{\mu=0}^{\infty} \frac{1}{\mu!} F^{(\mu)}(a)(x-a)^{\mu} \dots \qquad (2),$$

shall have a circle of convergence, is satisfied.

In the theory of analytic functions constructed by Weierstrass, the function is defined by the series $P(x|a)$ and by the analytic continuation of this series. The function is completely determinate provided the elements

$$F(a),\ F^{(1)}(a),\ F^{(2)}(a), \dots,\ F^{(\mu)}(a), \dots$$

are given. We denote generally by $F(x)$ the function in its totality which is defined by these elements.

If $K$ is a continuum formed by a single piece, which nowhere overlaps itself and encloses the point $a$, and if it is such that the branch of the function $F(x)$ formed by $P(x|a)$ and by its analytic continuation within $K$ remains uniform and regular, I shall denote this branch by $FK(x)$. The problem to be discussed here is that of finding

* Cauchy, *Cours d'Analyse de l'École royale polytechnique*, 1ère partie, Analyse Algébrique, Paris 1821, chapitre 9, § 2, theorème I, p. 286. Expressed in modern phraseology, Cauchy's condition would be formulated thus: *The upper limit of the limiting values of the moduli*

$$\left| \left\{ \frac{1}{\mu!} F^{(\mu)}(a) \right\}^{\frac{1}{\mu}} \right|$$

*is a finite magnitude.* It is known that, if this finite magnitude be denoted by $\frac{1}{r}$, the quantity $r$ is the radius of the circle of convergence of the series (2).

*an analytical representation of a branch $FK(x)$ which is to be chosen as extensive as possible.*

Merely from the definition of the analytic function $F(x)$ and from that of the branch $FK(x)$, there follows at once a kind of analytical representation of the branch $FK(x)$ in question. In effect, such a representation is always given by an enumerable number of analytical continuations of $P(x|a)$. But as the radius of the circle of convergence of such an analytical continuation is given only by Cauchy's criterion already quoted, this mode of representing $FK(x)$ becomes extremely complicated and rather unworkable. The analytical continuation ought rather to be regarded as the definition of the function than as a mode of representation.

There is another mode of representation which arises immediately from the principles upon which Cauchy's theory of functions is based. Such a representation is given by the formula

$$FK(x) = \int^S \frac{FK(z)}{z-x} dz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where the integral is taken along a closed contour $S$ within $K$. By the definition of an integral, it is clear that the integral (3) can be replaced by an infinite sum of rational functions of $x$, the coefficients of which are expressed by special values of $x$ (there being an enumerable number of these) and the corresponding values of $FK(x)$. This observation was the point of departure of the investigation of M. Runge* as well as of the subsequent investigations of MM. Painlevé, Hilbert and others. The analytical representation thus obtained accordingly requires a knowledge of the value of $FK(x)$ at an infinite and enumerable number of points. Now in the customary problems of analysis these values are not known. In general it is, on the contrary, the series of values

$$F(a),\ F^{(1)}(a),\ F^{(2)}(a), \ldots$$

which is given. Adopting the usual point of view, it is thus for instance in the problem of the integration of differential equations.

When, then, we have to find the analytical representation of $FK(x)$, it must be drawn from the elements (1) and, by means of those elements alone, a formula must be constructed to represent the branch $FK(x)$ completely. Let $C$ denote the circle of convergence of the series (2). The expression

$$\sum_{\mu=0}^{\infty} \frac{1}{\mu!} F^{(\mu)}(a)(x-a)^\mu$$

then gives the analytical representation of $FC(x)$, the equality

$$FC(x) = \sum_{\mu=0}^{\infty} \frac{1}{\mu!} F^{(\mu)}(a)(x-a)^\mu$$

holding for all points within $C$. This expression is constructed by means of the elements

$$F(a),\ F^{(1)}(a),\ F^{(2)}(a), \ldots$$

* "Zur Theorie der eindeutigen analytischen Functionen," § 1, pp. 229—239, *Acta Mathematica*, tome 6.

and of the rational numbers $\frac{1}{\mu!}$ independent of the choice of the elements: and it is to be remarked that the expression is formed without any *a priori* knowledge of the radius of the circle $C$. This radius is determinate, in connection with the elements, by Cauchy's theorem, and there are various methods of obtaining it from them; but it does not enter explicitly into the expression. Thus Taylor's series is formed simply by the elements

$$F(a),\ F^{(1)}(a),\ F^{(2)}(a), \ldots,$$

when these are the derivatives of the function.

The following question may therefore be proposed: Is it possible to obtain for a branch $FK(x)$ with the greatest range possible an analytical representation of this nature? As I have shewn in various notes, published in Swedish by the Stockholm Academy of Sciences during the past year, the reply is in the affirmative, and consequently it is possible to fill an important lacuna in the theory of analytic functions. In fact, hitherto it has been impossible to give for the general branch $FK(x)$ an analytical representation similar to that found from the very beginning of the theory for the branch $FC(x)$.

For a fundamental treatment of the question which has been proposed, it is first necessary to define a domain $K$ which shall be as great as possible. This I shall do by the introduction of a new geometrical conception—a *Star*-figure.

In the plane of the complex variable $x$, let an area be generated as follows. Round a fixed point $a$ let a vector $l$ (a straight line terminated at $a$) revolve once: on each position of the vector, determine uniquely a point, say $a_l$, at a distance from $a$ greater than a given positive quantity, this quantity being the same for all positions of the vector. The points thus determined may be at a finite or at an infinite distance from $a$. When the distance between $a_l$ and $a$ is finite, the part of the vector from $a_l$ to infinity is excluded from the plane of the variable.

The region which remains after all these sections (*coupures*) in the plane of $x$ have been made is what I call a *Star*-figure. Manifestly the star is a continuum formed of a single simply-connected area.

Associate with $a$ the elements

$$F(a),\ F^{(1)}(a),\ F^{(2)}(a), \ldots,\ F^{(\mu)}(a), \ldots$$

satisfying Cauchy's condition; and form the series

$$P(x|a) = \sum_{\mu=0}^{\infty} \frac{1}{\mu!} F^{(\mu)}(a)(x-a)^{\mu}.$$

Construct the analytical continuation of $P(x|a)$ along a vector from $a$. It may be the case that every point of this vector belongs to the circle of convergence of a series which itself is an analytical continuation of $P(x|a)$ obtained by proceeding along the vector; but it is also possible that, on proceeding along the vector, a point is met not situated within the circle of convergence of any analytical continuation of $P(x|a)$ along the vector. In the latter case, I exclude from the plane of the variable that part of the vector comprised between the point thus met and infinity. On making this vector describe one complete revolution round $a$, a *Star*-figure (as defined above) is obtained.

This star being given in a unique manner as soon as the elements (1) are assigned, I call it the *Star belonging to these elements*, and I denote* it by $A$. In defining the star, straight lines have been used as vectors: it is easy to see that curved lines, suitably defined, might have been chosen for the purpose.

In accordance with the phrase *the star belonging to the elements* (1), I speak of *the function* $F(x)$, as well as of *the functional branch* $FA(x)$, *belonging to these elements.*

These preliminaries being settled, my main theorem is as follows:—

*Denote by $A$ the star belonging to the elements*

$$F(a),\ F^{(1)}(a),\ F^{(2)}(a),\ \ldots\ldots$$

*and by $FA(x)$ the corresponding functional branch belonging to the same elements; let $X$ be any finite domain within $A$; and let $\sigma$ denote a positive quantity as small as we please. Then it is always possible to find an integer $\bar{n}$ such that the modulus of the difference between $FA(x)$ and the polynomial*

$$g_n(x) = \sum_\nu c_\nu^{(n)} F^{\nu}(a)(x-a)^\nu$$

*for values of $n$ greater than $\bar{n}$, is less than $\sigma$ for all the values of $x$ belonging to $X$. The coefficients $c_\nu^{(n)}$ are chosen a priori and are absolutely independent of $a$, of $F(a)$, $F^{(1)}(a)$, $F^{(2)}(a)$, ..., and of $x$.*

It is very important to observe that the explicit knowledge of the star is not necessary for the construction of the function $g_n(x)$. When the elements $F(a)$, $F^{(1)}(a)$, $F^{(2)}(a)$, ... are once given, the star belonging to them is definite; but it does not enter explicitly into the expression $g_n(x)$. The case is precisely the same as for Taylor's series where the radius of the circle of convergence does not enter explicitly into the expression.

The theorem can be proved by very elementary considerations, using especially the fundamental theorem established by Weierstrass in his memoir *Zur Theorie der Potenzreihen*, dated† 1841.

Passing from the same theorem for functions of several variables, we can easily obtain a generalisation of my main theorem which includes the case of any finite number of independent variables.

The coefficients denoted by $c_\nu^{(n)}$ are given *a priori*. They are quite independent of the special function to be represented just as are the coefficients $\frac{1}{\mu!}$ in Taylor's series. But the choice of these coefficients is not unique. On the contrary it can be made in an infinitude of ways; and when conditions are given, the problem arises of making a choice which is the best adapted to these conditions.

* As the first letter of the word ἀστήρ.

† *Ges. Werke*, Bd. 1, p. 67.

The formula

$$g_n(x) = \sum_{h_1=0}^{n^2} \sum_{h_2=0}^{n^4} \cdots \sum_{h_n=0}^{n^{2n}} \frac{1}{h_1!\, h_2! \ldots h_n!} F^{(h_1+h_2+\ldots+h_n)}(a) \left(\frac{x-a}{n}\right)^{h_1+h_2+\ldots+h_n} \quad \ldots\ldots\ldots\ldots(4)$$

gives an expression for $g_n(x)$ which perhaps is the simplest of all as regards the mere form. There are other forms in which the coefficients $c_\nu^{(n)}$ are rational numbers, or are numbers depending in a special manner upon the transcendents $e$ and $\pi$, and which are of great simplicity.

Upon this I shall not dwell: but I enunciate another theorem which is an almost immediate consequence of my main theorem.

*Denote by $A$ the star which belongs to the elements*

$$F(a),\ F^{(1)}(a),\ F^{(2)}(a), \ldots\ldots,$$

*and by $FA(x)$ the corresponding functional branch belonging to the same elements. This branch $FA(x)$ can always be represented by a series*

$$\sum_{\mu=0}^{\infty} G_\mu(x),$$

*where the quantities $G_\mu(x)$ are polynomials of the form*

$$G_\mu(x) = \sum_\nu \mathfrak{c}_\nu^{(\mu)} F^{(\nu)}(a)(x-a)^\nu,$$

*each coefficient $\mathfrak{c}_\nu^{(\mu)}$ being a determinate number (which can be taken as rational) depending only upon $\mu$ and $\nu$. The series*

$$\sum_{\mu=0}^{\infty} G_\mu(x),$$

*converges for every value of $x$ within $A$, and it converges uniformly for every domain within $A$. For all values within $A$ we have*

$$\sum_{\mu=0}^{\infty} G_\mu(x) = \operatorname*{Lim}_{n=\infty} g_n(x),$$

*where $g_n(x)$ is the polynomial in my main theorem.*

In what precedes, a definition has been given of the *star belonging to the elements*

$$F(a),\ F^{(1)}(a),\ F^{(2)}(a), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

In accordance with this terminology, we can speak of *the circle belonging to the elements* (1) which, in fact, is the circle of convergence $C$ of the series

$$P(x \mid a) = \sum_{\mu=0}^{\infty} \frac{1}{\mu!} F^{(\mu)}(a)(x-a)^\mu.$$

It is evident that this circle is inscribed in the star which belongs to the same elements. The circle may be regarded as a first approximation to the star. To the circle $C$ corresponds an analytical expression $P(x \mid a)$ which has the property of representing $FA(x)$ within $C$, of converging uniformly for any domain within $C$, and of ceasing to converge outside

$C$. Between the circle and the star, intermediary domains $C^{(\mu)}$, $(\mu=1, 2, 3, \ldots)$, exist, unlimited in number; each of them in succession includes the domain that precedes it; and they can be chosen so that, corresponding to each domain $C^{(\mu)}$, there is an analytical expression representing $FA(x)$ within $C^{(\mu)}$ which converges uniformly for every domain within $C^{(\mu)}$ and ceases to converge outside $C^{(\mu)}$. On this question there is an interesting study to be made which I have merely sketched in my Swedish memoirs; to it I shall return on another occasion.

The only writer who, so far as I know, has found a general representation of $FA(x)$ valid outside the circle belonging to the elements (1) is M. Borel. In two important memoirs*, M. Borel is concerned with what he calls the summability of a series. It appears to me that the chief interest of this investigation of M. Borel is that the author really finds an expression valid for a domain which in general includes the circle $C$. The domains which I have called $C^{(\mu)}$ can easily be chosen so that $C^{(1)}$ becomes this domain $K$: so that M. Borel's domain $K$ becomes the second approximation to the Star, the circle being the first as already indicated.

But M. Borel has discussed the same class of ideas in another publication. In his book† published without any acquaintance with my Swedish Notes of the same year, the author says‡:—

"Pour résumer les résultats acquis sur le problème de la représentation analytique "des fonctions uniformes, nous pouvons dire§ que nous en connaissons deux solutions "complètes; l'une est fournie par le théorème de Taylor, l'autre par le théorème de "M. Runge‖. Ces deux solutions ont une très grande importance à cause de leur "généralité; mais chacune d'elles a de graves inconvénients dont les principaux sont, pour "la série de Taylor, de diverger en des régions où la fonction existe; et, pour la repré-"sentation de M. Runge et celles de M. Painlevé, d'être possibles d'une infinité de "manières¶.

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"Le but idéal à atteindre, c'est de trouver une représentation réunissant les avantages "de la série de Taylor et des séries de M. Runge ou de M. Painlevé, sans avoir aucun "de leurs inconvénients**, et le but immédiat, c'est de trouver une telle représentation "pour des classes de fonctions de plus en plus étendues††."

* *Journal de Mathématiques*, 5me Sér., t. ii. (1896), "Fondements de la théorie des séries divergentes sommables," pp. 103—122; "Sur les séries de Taylor admettant leur cercle de convergence comme coupure," pp. 441—454.

† *Leçons sur la théorie des fonctions*, Paris, 1898.

‡ pp. 88 ff.

§ All that follows on the analytical representation of uniform functions can be applied, *mutatis mutandis*, to the functional branch $FA(x)$.

‖ I have indicated above that, in M. Runge's theorem, there is nothing which is not already in principle contained in the representation by Cauchy's integral.

¶ In what precedes, I have pointed out what appears to me a graver inconvenience, viz. that these expressions require the knowledge of an enumerable number of values of the function which correspond to points that approach indefinitely near the limit of existence of the function.

** It will be seen that I have achieved this aim, not only for uniform analytic functions but also for the functional branch $FA(x)$. It might be asked whether it would not be possible to achieve the same aim for the function $F(x)$ in its totality. It is not so: such a question is too general. The problem was mainly that of limiting the question so as to make a solution possible without diminishing the generality more than was necessary. I believe that this problem is solved by the introduction of the star and of the functional branch $FA(x)$.

†† It appears that M. Borel has not regarded his own

There exist a certain number of other investigations having relations with my theorems but belonging to a range of ideas quite different from M. Borel's. I have already spoken of the representation which follows from Cauchy's integral

$$FA(x) = \frac{1}{2\pi i}\int^{S} \frac{FA(z)}{z-x}\,dz.$$

With M. Runge, we can transform this integral into a series every term of which is a polynomial in $x$. But in order to construct these polynomials, it is necessary to know not only the star $A$ but also the values of the function for an enumerable number of points approaching indefinitely near the boundary of $A$. Investigations have been carried out in which the elements $F(a)$, $F^{(1)}(a)$, $F^{(2)}(a)$, ... are substituted for these values of the function. But these investigations always abut, in a manner more or less direct, upon the conformal representation of the circle of convergence on another figure known beforehand: and they still require that we should know, as to the function which is to be represented, that it is regular within the domain represented on the circle. The most interesting and the most significant theorem in this range of ideas appears to me to be that of M. Painlevé*:

*Given a convex domain $D$ and an internal point $a$, a set of polynomials*

$$\Pi_{\mu 0}(x),\ \Pi_{\mu 1}(x), \ldots,\ \Pi_{\mu\mu}(x);\ (\mu = 1, 2, 3, \ldots),$$

*can be constructed such that any function $F(x)$ holomorphic in $D$ is developable in that domain in the form*

$$F(x) = \sum_{\mu=0}^{\infty} \{F_0(a)\,\Pi_{\mu 0}(x) + F^{(1)}(a)\,\Pi_{\mu 1}(x) + \ldots + F^{(\mu)}(a)\,\Pi_{\mu\mu}(x)\}.$$

The resemblance between M. Painlevé's formula and mine is obvious. Writing

$$\Pi_{\mu\nu}(x) = \mathfrak{c}_{\nu}^{(\mu)}(x-a)^{\nu}$$

in M. Painlevé's formula, mine follows. Yet the resemblance is entirely formal, because the formation of the polynomials $\Pi_{\mu 0}(x)$, $\Pi_{\mu 1}(x)$, ... , $\Pi_{\mu\mu}(x)$ requires the *a priori* establishment of the domain $D$ and the knowledge of the function $F(x)$ that it is holomorphic in $D$: whereas with me the formula of representation, so far from supposing any *a priori* knowledge of the star $A$, gives on the contrary the means of determining the star†.

In other publications, it is my intention to develop other theorems in the same range of ideas as well as to return to the numerous applications that can be made of my theorems: I restrict myself in this place to the following indications. I have just explained that, besides the circle $C$ and the star $A$, there is an infinite number of other

investigations on the summability of series from the point of view just indicated so clearly in his book. Otherwise he rather might have said: that the immediate aim was to find a general representation valid for a domain still more extensive than this domain $K$ (that is, $C^{(1)}$).

* *Comptes Rendus*, t. cxxvi (24 Jan., 1898), pp. 320, 321.

† While the present note was passing through the press, a new and interesting note of M. Painlevé's, discussing the relation of these investigations to my own, has appeared in the *Comptes Rendus* (23 May, 1899). In the same number of the *Comptes Rendus*, there is a note by M. Borel related to my investigations. The reader is also referred to an addition to the "mémoire sur les séries divergentes par É. Borel" (*Ann. de l'Éc. Norm.*, 1899), and to two important notes by M. Picard (*Comptes Rendus*, 5 June, 1899) and M. Phragmén (*Comptes Rendus*, 12 June, 1899): all of them are connected with these investigations.

stars $C^{(1)}$, $C^{(2)}$, $C^{(3)}$, ... each of which is circumscribed to that which precedes it and is inscribed* to that which follows it; to these there correspond expansions $PC^{(1)}(x|a)$, $PC^{(2)}(x|a)$, $PC^{(3)}(x|a)$, ... which preserve all the principal characters of the Taylor's series $PC(x|a)$. The expression $PC^{(\mu)}(x|a)$ is merely a $(\mu+1)$-ple series with limited convergence.

There is another method of generalising Taylor's series as follows:

*Denote by $A$ a star with its centre at $a$, and by $A^{(\delta)}$ an associate star, concentric with $A$ and inscribed in $A$, defined with reference to $A$ in some suitable manner. This star $A^{(\delta)}$ is to be such that it becomes a circle when $\delta=1$ and that it encloses in its interior every domain within $A$ when the quantity $\delta$ is sufficiently small.*

*Now suppose that $A$ is the star belonging to the elements $F(a)$, $F^{(1)}(a)$, $F^{(2)}(a)$, ..., and construct the series*

$$P_\delta(x|a)=F(a)$$
$$+\sum_{\lambda=1}^{\infty}\{h_1^{(\lambda)}(\delta)F^{(1)}(a)(x-a)+h_2^{(\lambda)}(\delta)F^{(2)}(a)(x-a)^2+\ldots+h_\lambda^{(\lambda)}(\delta)F^{(\lambda)}(a)(x-a)^\lambda\} \quad \ldots(5).$$

*The coefficients*

$$h_\mu^{(\lambda)}(\delta), \qquad \begin{pmatrix}\mu=1,\ 2,\ldots,\ \lambda\\ \lambda=1,\ 2,\ldots,\ \infty\end{pmatrix},$$

*can be assigned a priori, independently of $a$, of $F(a)$, $F^{(1)}(a)$, $F^{(2)}(a)$, ... and of $x$, so that the series possesses the following properties: it converges for every point within $A^{(\delta)}$ and converges uniformly for every domain within $A^{(\delta)}$. If convergence takes place for any value, the value necessarily belongs to the interior of $A^{(\delta)}$ or is a point of the star $A^{(\delta)}$. When $\delta=1$, the series becomes Taylor's series.*

*The equality*

$$FA(x)=P_\delta(x|a),$$

*exists throughout the interior of $A^{(\delta)}$.*

Among other differences between the two generalisations of Taylor's theorem, this may be noted: that in the first the stars $C^{(1)}$, $C^{(2)}$, $C^{(3)}$, ... form, so to speak, a discontinuous sequence of domains of convergence, while in the second there is a continuous transition from the circle $C(=A^{(1)})$ to the star $A\ (=A^{(0)})$.

The star which belongs to the elements $F(a)$, $F^{(1)}(a)$, ... is given at the same time as these elements, just as the circle which belongs to the elements also is given. But in order actually to construct the star on the circle, we must in the first case know the points of the star (it is thus that I describe the points formerly denoted by $a_l$) and in the second case the distance between $a$ and the nearest point of the star. It might be difficult to deduce this knowledge simply by the study of the elements $F(a)$, $F^{(1)}(a)$, $F^{(2)}(a)$, .... But in some problems the points of the star are directly given: e.g. the determination of the general integral of a differential all of whose critical points are fixed, being finite in number. In this case, we can construct the star directly and can obtain an analytical expression for the integral valid over the whole plane except

* A star is inscribed in another which circumscribes it if the whole of the first star belongs to the second and if the two stars have common points such as $a_l$.

at a finite number of determinate sections. Notwithstanding the remarkable researches of M. Fuchs and M. Appell and others, this problem of finding a representation, which at once is unique for the whole plane and is sufficiently simple, has not hitherto been solved.

The beautiful researches of MM. Fabry, Hadamard, Borel and other French writers, which have their origin in M. Darboux's memoir* "Sur l'approximation des fonctions de très-grands nombres" and which aim at the development of the criteria whether a point on a circle belonging to the elements $F(a)$, $F^{(1)}(a)$, $F^{(2)}(a), \ldots$ is a singularity of the function or not, are well known. My theorems make it possible to study this problem from a more general point of view than these writers and to find the criteria which distinguish the points of the star belonging to the elements $F(a)$, $F^{(1)}(a)$, $F^{(2)}(a), \ldots$ from other points. It can be stated that, to each selection of the coefficients called $c_\nu^{(n)}$, there corresponds a special system of criteria.

For these investigations, the following theorem can serve as the point of departure:—

*If $x$ is a point within the star $A$ belonging to the elements $F(a)$, $F^{(1)}(a)$, $F^{(2)}(a), \ldots$, and if $\epsilon$ is a positive quantity sufficiently small, it is always possible to choose a positive number $\delta$ so that, $\sigma$ being a positive quantity as small as we please, a positive integer $\bar{\lambda}$ exists such that*

$$|h_1^{(\lambda)}(\delta) F^{(1)}(a)(1+\epsilon)(x-a) + h_2^{(\lambda)}(\delta) F^{(2)}(a)\{(1+\epsilon)(x-a)\}^2 + \ldots + h_\lambda^{(\lambda)}(a) F^{\lambda}(a)\{(1+\epsilon)(x-a)\}^\lambda| < \sigma,$$

*provided*† $\lambda \geqslant \bar{\lambda}$.

*If on the contrary, $x$ does not lie within $A$, this property does not hold.*

M. Poincaré has pointed out a certain substitution which is of great value in the study of certain mechanical problems, particularly in that of $n$ bodies. When this substitution is used, a development of the function in powers of the time can be obtained which is valid for real values of the time as far as the first *positive* or *negative* singularity nearest the origin. But the mechanical problem requires in general a knowledge of the first positive singularity, and not merely the nearest singularity, positive or negative. It is obvious that the resolution of this problem can be brought within my theorem. In fact, knowing the elements $F(t_0)$, $F^{(1)}(t_0)$, $F^{(2)}(t_0), \ldots$ at a given epoch $t_0$, we can obtain a development which represents the function and is valid for all real values of $t > t_0$ up to the first singularity of the function.

Recently I had an opportunity of giving an account of a portion of my investigations before the Academy of Sciences of Turin. My friend M. Volterra then made the following interesting communication.

> If in any dynamical problem the unknown functions be regarded as analytic functions of the time, the problem will be solved completely from the analytical point of view when it can be shewn that the real axis of the time falls completely within the stars of the

* Liouville, *Journ. de Math.*, 3me Sér., t. iv. (1875), pp. 5—54.

† The quantities $\delta$ and $h_\mu^{(\lambda)}(\delta)$ have the same significance as in the formula (5).

unknown functions, the centre of the stars being the initial value of the time. In fact, it is sufficient to employ M. Mittag-Leffler's expansions to obtain the unknown functions for any value of the time. The coefficients in the expansions will be determined by the initial conditions of motion.

1°. A very extensive class of dynamical equations can be reduced to the integration of differential equations of the type

$$\dot{p}_s = \sum_1^\nu{}_s \sum_1^\nu{}_\kappa \, a_{s\kappa}^{(r)} p_\kappa p_r,$$

where $a_{s\kappa}^{(r)} + a_{s\kappa}^{(r)} = 0$. Since in this case a finite strip enclosing the real axis is contained in the stars of the functions $p_s$, the centre being $t = 0$, new forms of the integrals of these equations can be derivable by M. Mittag-Leffler's expansions*.

2°. Passing to the problems of attraction, it may be remarked that the problem of the motion of a point attracted by fixed points placed in a straight line, the force being according to Newton's law, has not been resolved when the number of attracting points is greater than two. Let us consider the general case and suppose that the moment of the initial velocity of the moving point $m$, with reference to the axis $x$ of fixed points, is not zero. Then $\vartheta$ being the angle which the plane $mx$ makes with a fixed plane through $x$, and $r$ being the distance of $m$ from the axis $x$, we have the areal integral

$$r^2\dot{\vartheta} = C = \text{constant},$$

and the integral of vis viva $T - P = h = \text{constant}$, where

$$T = \tfrac{1}{2}m\,(\dot{r}^2 + r^2\dot{\vartheta}^2 + \dot{x}^2), \quad P = \Sigma\,\frac{M_i m}{r_i},$$

$T$ being the vis viva and $P$ the potential: in the latter expression the masses of the fixed point are denoted by $M_i$ and their distances from $m$ by $r_i$. It is at once obvious that $r$ cannot vanish. In effect, if for $t = t_0$, $r$ can become indefinitely small, let us take this quantity as an infinitesimal of the first order. On account of the areal integral, $\dot{\vartheta}$ would be infinitely great of the second order, and consequently $r^2\dot{\vartheta}^2 \,(= C\dot{\vartheta})$ would also be of the second order: $T$ therefore would be infinitely great of the second order. But $P$ if it become infinitely great, can be so only to the first order because the quantities $r_i$ are greater than $r$; hence if $r$ could become infinitely small, the integral of vis viva would not be verified. It therefore is to be inferred that the real axis of the time is contained in the stars of the unknown elements: and consequently these elements are expressible by Mittag-Leffler's series.

3°. Given $n$ points repelling one another according to the Newtonian law of force, the integral of vis viva may be written

$$\tfrac{1}{2}\sum_i m_i\,(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) + \sum_{i,s}\frac{m_i m_s}{r_{i,s}} = h,$$

* I have studied this class of equations in three Notes published by the Academy of Turin in 1898 and 1899. The class can be still further extended so as to include many of the classical problems in dynamics.

where $x_i$, $y_i$, $z_i$ are the coordinates of the moving points, $m_i$ their masses, $r_{i,s}$ their distances, and $h$ is a constant quantity. By noting that in this equation all the terms are positive, we infer that the points cannot collide and that their velocities are finite. Hence in this case also, the real axis of the time lies within the stars. But we can pass from the case of repulsion to that of attraction by changing $t$ into $t\sqrt{-1}$. Through this transformation, the components of the velocities become imaginary if they were real, and *vice versa*. But if at the beginning of the time they were zero, the transformation leaves them zero. Hence we deduce the very curious theorem:

Consider the problem of $n$ bodies in the most general case, with the sole condition that the initial velocities of the bodies are zero: then taking the origin at the beginning of the time, the real axis is not included within the stars of the coordinates, but the imaginary axis is always completely included. That is to say, M. Mittag-Leffler's expansions will be valid for imaginary values of the time even if they are not so for all real values.

4°. Finally it may be remarked that M. Mittag-Leffler's expansions can be used for the motion of straight and parallel vortices. Reference may be made to Lecture XX. in Kirchhoff's *Mechanik* for the differential equations of the motion.

The interest of this development is manifest. I remark, however, that the main importance of my theorems so far as concerns mechanics appears to me to be that they provide a means of finding a real and positive point of my star, and of determining whether this point is at infinity or not. M. Volterra on the contrary supposes as always known beforehand that this point is at infinity. My principal theorem also provides in this case a means of representing the function, with any approximation desired for any real domain whatever, by a polynomial into which there enter no elements taken from the function other than a limited number of the quantities $F(t_0)$, $F^{(1)}(t_0)$, $F^{(2)}(t_0)$, .... It appears to me that this point of view may become useful in applications to mechanics.

PERUGIA, *April*, 1899.