Homework 4
Math 576

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Problem

Prove

$$H_+^{-1} = \left(I_n - \frac{sy^T}{s^T y}\right) H_c^{-1} \left(I_n - \frac{ys^T}{s^T y}\right) + \frac{ss^T}{s^T y} \tag{1}$$

where

$$H_+ = H_c + \frac{yy^T}{y^T s} - \frac{(H_c s)(H_c s)^T}{s^T H_c s}, \tag{2}$$

here$H_c \in \mathbb{R}^{n\times n}$ is invertible and symmetric, $y,\ s \in \mathbb{R}^{n\times 1}$ and I_n is the $n \times n$ identity matrix.

<u>Proof</u>

Equation (2) admits the following representation

$$H_+ = H_c + CD^T, \tag{3}$$

where[1]

$$C = [y \quad H_c s]\,,\ D^T = \begin{bmatrix} \frac{y^T}{y^T s} \\ -\frac{(H_c s)^T}{s^T H_c s} \end{bmatrix}, \tag{4}$$

if we apply the Sherman-Morrison-Woodbury formula to (3) we obtain

$$H_+^{-1} = H_c^{-1} - H_c^{-1} C \left(I_2 + D^T H_c^{-1} C\right)^{-1} D^T H_c^{-1}, \tag{5}$$

after substituting (4) in (5) we obtain

$$H_+^{-1} = H_c^{-1} - P^T Q R, \tag{6}$$

[1] C and D are $n \times 2$ matrices represented in block form.

where[2]

$$
\begin{aligned}
P^T &= H_c^{-1}\,[y \;\; H_c s]\,,\\
Q &= \left(I_2 + \begin{bmatrix} \frac{y^T}{y^T s} \\ -\frac{(H_c s)^T}{s^T H_c s} \end{bmatrix} H_c^{-1}\,[y \; H_c s]\right)^{-1},\\
R &= \begin{bmatrix} \frac{y^T}{y^T s} \\ -\frac{(H_c s)^T}{s^T H_c s} \end{bmatrix} H_c^{-1}.
\end{aligned}
$$

The matrices P^T, Q and R can be simplified to

$$
\begin{aligned}
P^T &= \left[H_c^{-1} y \;\; s\right],\\
Q &= \frac{s^T H_c s}{s^T y}\begin{pmatrix} 0 & -1 \\ \frac{s^T y}{s^T H_c s} & 1 + \frac{y^T H_c^{-1} y}{y^T s} \end{pmatrix},\\
R &= \begin{bmatrix} \frac{y^T H_c^{-1}}{y^T s} \\ -\frac{s^T}{s^T H_c s} \end{bmatrix},
\end{aligned}
$$

where property $H_c^T = H_c$ was used. Now, after doing some block-matrix multiplication we simplify the product $P^T QR$ to

$$
\begin{aligned}
P^T QR &= \frac{s^T H_c s}{s^T y}\left[\frac{s^T y}{s^T H_c s}\frac{s y^T H_c^{-1}}{y^T s} + \frac{H_c^{-1} y s^T}{s^T H_c s} - \left(1 + \frac{y^T H_c^{-1} y}{y^T s}\right)\frac{s s^T}{s^T H_c s}\right]\\
&= \frac{s y^T H_c^{-1}}{s^T y} + \frac{H_c^{-1} y s^T}{s^T y} - \frac{s s^T}{s^T y} - \frac{y^T H_c^{-1} y}{(s^T y)^2} s s^T,
\end{aligned}
$$

then (6) takes the form

$$
\begin{aligned}
H_+^{-1} &= H_c^{-1} + \frac{s s^T}{s^T y} + \frac{y^T H_c^{-1} y}{(s^T y)^2} s s^T - \frac{s y^T H_c^{-1}}{s^T y} - \frac{H_c^{-1} y s^T}{s^T y}\\
&= H_c^{-1}\left(I_n - \frac{y s^T}{s^T y}\right) + \frac{y^T H_c^{-1} y}{(s^T y)^2} s s^T - \frac{s y^T H_c^{-1}}{s^T y} + \frac{s s^T}{s^T y},
\end{aligned}
$$

but[3] $\frac{y^T H_c^{-1} y}{(s^T y)^2} s s^T = \frac{s y^T H_c^{-1} y s^T}{(s^T y)^2}$, therefore

$$
\begin{aligned}
H_+^{-1} &= H_c^{-1}\left(I_n - \frac{y s^T}{s^T y}\right) + \frac{s y^T H_c^{-1} y s^T}{(s^T y)^2} - \frac{s y^T H_c^{-1}}{s^T y} + \frac{s s^T}{s^T y}\\
&= H_c^{-1}\left(I_n - \frac{y s^T}{s^T y}\right) - \frac{s y^T H_c^{-1}}{s^T y}\left(I_n - \frac{y s^T}{s^T y}\right) + \frac{s s^T}{s^T y} \qquad (7)\\
&= \left(I_n - \frac{s y^T}{s^T y}\right) H_c^{-1}\left(I_n - \frac{y s^T}{s^T y}\right) + \frac{s s^T}{s^T y},
\end{aligned}
$$

[2]It's clear that $P^T \in \mathbb{R}^{n\times 2}$, $Q \in \mathbb{R}^{2\times 2}$ and $R \in \mathbb{R}^{2\times n}$.

[3]Indeed, $y^T H_c^{-1} y$ is a scalar and it can be located between s and s^T, the final product is consistent respect to matrix dimensions.

where $H_c^{-1}\left(I_n - \frac{ys^T}{s^Ty}\right)$ was factored to the right of the first two terms of (7). Therefore (1) is true provided H_c is invertible and symmetric.