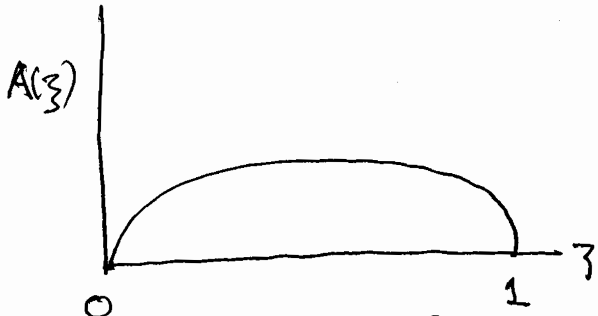
while $z^2 = A^3$ & $\int_0^1 A \, d\xi = V/2$

gives
$$\left.\begin{aligned} A &= \frac{2V}{\pi L}\sin^2\vartheta(\xi) \\ \xi &= \frac{1}{\pi}\left(\vartheta - \tfrac{1}{2}\sin 2\vartheta\right) \end{aligned}\right\} \quad (24)$$

$\left(C^{2/3} = \frac{2V}{\pi L}\right) = \frac{2}{\pi}\sqrt{\frac{\lambda}{3}}$

i.e. $\lambda = 3V^2/L^2 = \dfrac{PL^2}{\alpha E}$ (25)



i.e. $P \leq \cancel{4\pi}\; 3\alpha_2 E \dfrac{V^2}{L^4}$ ($\alpha_2 = 1$ for symmetric section)

For right circular column $P \leq \dfrac{2\pi^2}{9\sqrt{3}} \dfrac{V^2 E}{L^4}$

<u>Example</u> Eigenvalue optimization

(1) $u_{xx} + (\lambda + V(x))u = 0$

(2) $\int_{-\infty}^{\infty} V^n(x)\, dx = K$

Find potentials $V(x)$ which make stationary some eigenvalue λ of (1) among all potentials satisfying (2).

Suppose $V_0(x)$ is a solution & suppose λ_0 & $u_0(x)$ constitute a stationary eigenvalue & eigenfunction. Introduce $V(x,\varepsilon)$ satisfying (2), and corresp. efunc. $u(x,\varepsilon)$ & eval. $\lambda(\varepsilon)$. Since $\lambda_0 = \lambda(0)$, $\Rightarrow$ $\dot{\lambda}^*(0) = 0$ (extremum).

Then

$$(3) \quad \dot{u}^*_{0,xx} + \left(\lambda_0 + V_0(x)\right)\dot{u}_0^* = -\dot{V}^*(x,0)\,u_0$$

$$(4) \quad \int_{-\infty}^{\infty} V_0^{n-1}\,\dot{V}^*(x,0)\,dx = 0$$

Fredholm alternative (u_0 solves same problem, equ. (1))
self-adjoint

$$\int_{-\infty}^{\infty} u_0\left(\dot{u}_{0xx} + (\lambda + V_0)\dot{u}_0\right)dx \qquad \left(u_0 \xrightarrow{x\to\pm\infty} 0\right.$$

$$= u_0\dot{u}_{0x}\Big|_{-\infty}^{\infty} - u_{0,x}\dot{u}_0\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\dot{u}_0\underbrace{\left[u_{0,xx} + (\lambda + V_0)u_0\right]}_{=0}dx$$

$$= -\int_{-\infty}^{\infty}\dot{V}(x,0)\,u_0^2(x)\,dx = 0 \quad \text{for all admissible } \dot{V}.$$

$$\therefore \quad u_0^2(x) = \overset{\text{set to 1,}}{\overset{\|}{\text{Const}}} * V_0^{n-1}(x)$$

Then (1) becomes

$$(7) \quad u_{0,xx} + u_0^{1+\frac{2}{n-1}} + \lambda_0 u_0 = 0 \qquad (\text{mult. by } u_{0x}) \Rightarrow$$

$$\Rightarrow \quad (8) \quad u_{0,x}^2 + \left(\frac{n-1}{n}\right)u_0^{\frac{2n}{n-1}} + \lambda_0 u_0^2 = 0 \qquad \left(= C, \text{ but let } x \to \pm\infty, \text{ get } C = 0\right)$$

$$\Rightarrow \quad u_{0,x} = (-\lambda_0)^{1/2} u_0 \left[1 + \frac{n-1}{n\lambda_0} u_0^{2/n-1} \right]^{1/2}$$

replacing u_0 by v_0, $u_0 = v_0^{\frac{n-1}{2}}$, we have

$\cancel{v_{0,x}}$

$$\underbrace{2u_0 u_{0x}}_{(n-1)v_0^{n-2}v_{0,x}} = 2(-\lambda_0)^{1/2} \overbrace{u_0^2}^{v_0^{n-1}} \left[1 + \frac{n-1}{n\lambda_0} v_0 \right]$$

$$\Rightarrow \quad v_{0,x} = \frac{2(-\lambda_0)^{1/2}}{n-1} v_0 \left[1 + \frac{n-1}{n\lambda_0} v_0 \right]^{1/2} \qquad (8')$$

The solutions of (8') differ by translations (since x does not enter explicitly). Select the solution for which

$$v(0) = n/n-1$$

to find

$$\frac{2(-\lambda_0)^{1/2}}{n-1} x = \int_{n/n-1}^{v_0} v^{-1} \left(1 + \frac{n-1}{n\lambda} v \right)^{-1/2} dv$$

$$= -2 \tanh^{-1} \left(1 + \frac{n-1}{n\lambda} v_0 \right)^{1/2}$$

$$\Rightarrow \quad v_0(x) = \frac{-n\lambda_0}{n-1} \operatorname{sech}^2 \left[\frac{(-\lambda_0)^{1/2} x}{n-1} \right] \qquad (9)$$

Therefore, $v_0(x)$ is periodic in x if $\lambda_0 > 0$ and $v_0(x) \equiv 0$ if $\lambda_0 = 0$. But integral (2) would not exist if $v_0(x)$ periodic and no evalue would exist if $v_0 \equiv 0$.

$\therefore\ \lambda_0 < 0 \quad (\Rightarrow V_0(x) \to 0 \text{ as } x \to \pm\infty)$

Now, $\left.\begin{array}{l} V_0(x) \to 0 \text{ as } x \to \pm\infty \\ U_0(x) \to 0 \text{ as } x \to \pm\infty \end{array}\right\} \Rightarrow$ our analysis is valid only for $n>1$ (from (6))

Equs. (6) & (9) $\Rightarrow$

(10) $$U_0(x) = \left[\frac{(-\lambda_0)n}{n-1}\,\mathrm{sech}^2\left(\frac{(-\lambda_0)^{1/2}x}{n-1}\right)\right]^{\frac{n-1}{2}} \quad (\neq 0)$$

$\therefore\ \lambda_0$ is lowest evalue (no internal zeros)

(standard Sturm-Liouville theory shows if lowest evalue is simple, analysis valid).

Substituting (9) into (2):

(11) $$\left(\frac{-n\lambda_0}{n-1}\right)^n \frac{2(n-1)}{(-\lambda_0)^{1/2}} \int_0^\infty \mathrm{sech}^{2n} y\, dy = k$$

$$\left(\int_0^\infty \mathrm{sech}^{2n} y\, dy = \frac{\Gamma(n)\Gamma(1/2)}{2\Gamma(n+\frac{1}{2})}\right)$$

(12) $$\therefore\ -\lambda_0 = -F(n)\,k^{\frac{2}{2n-1}}$$

$(n>1,\ \lambda_0<0 \Rightarrow k>0)$

where $$F(n) = \left[\frac{\Gamma(n+\frac{1}{2})}{(n-1)\sqrt{\pi}\,\Gamma(n)}\left(\frac{n-1}{n}\right)^n\right]^{2/2n-1}$$

∴ For any $k>0$ and any $n>1$ $\exists$ one potential $V_0(x)$ given by (9) with λ_0 given by (12) which makes the lowest evalue of (1) stationary. Assume λ_0 is <u>min</u> value of lowest evalue (did not show this!).

Then if λ is the lowest evalue of some potential $V(x)$, we have $\lambda \geq \lambda_0$.

Potential $V(x)$ satisfies (2). If we define k in terms of V by (2) and if we use (12) for λ_0, we obtain

(13) $$\lambda \geq -F(n)\left[\int_{-\infty}^{\infty} V^n(x)\,dx\right]^{2/2n-1} \quad \text{for each } n>1$$

equality here for $V(x) = V_0(x)$.

<u>Note</u>: As $n \to 1$, (13) becomes

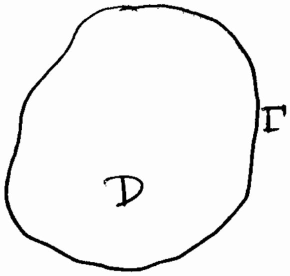
(14) $$\lambda \geq -\frac{1}{4}\left[\int_{-\infty}^{\infty} V(x)\,dx\right]^{1/2}$$

And (9) $\Rightarrow V_0(x) \to \delta(x)$ with equality holding in (14). Of all $V(x)$ of given area, the δ-function binds the best, that is the smallest $E<0$ (or largest $-E$) occurs for δ-function potential.

J. Keller, J. Math. Phys. 2 (1961) 262-266 (see for proof that λ_0 is minimum).

The Elastic Plate: Hamilton's Principle

$$\delta \int_{t_0}^{t_1} (T-V)\,dt = 0$$



$u(x,y,t)$: transverse deflection

ρ : mean density

$$T = \iint_D \tfrac{1}{2}\rho u_t^2 \,dx\,dy$$

V = (potential energy of deformation) V_1

+ (potential energy due to bending moments of density $m(s,t)$ prescribed on Γ) V_2

+ (p.e. due to external forces on surface, D, of density $f(x,y,t)$ and on Γ with linear density $p(s,t)$) V_3

Poisson's ratio, usually $\in (0, \frac{1}{2})$

$$= \frac{1}{2} c \iint_D \Big[(u_{xx}+u_{yy})^2 - 2(1-\mu)\underbrace{(u_{xx}u_{yy}-u_{xy}^2)}_{=\nabla\cdot(u_x u_{yy},\, -u_x u_{xy})}\Big]\,dx\,dy$$

c, μ constants

so this term will not enter in the equations, but ~~other~~ nevertheless plays crucial role

$$+ \oint_\Gamma m(s,t)\frac{\partial u}{\partial n}\,ds$$

$$+ \iint_D f u \,dx\,dy + \oint_\Gamma p(s,t)\,ds$$

Variation : $u \longrightarrow u + \varepsilon\eta$

Let $\Psi(\varepsilon) = \int_{t_0}^{t_1} \left[T(u+\varepsilon\eta) - V(u+\varepsilon\eta)\right] dt$

$$\dot{\Psi}(0) = 0 = \int_{t_0}^{t_1} \iint_D \left(-\rho u_{tt} - \nabla^4 u - f\right)\eta \,dx\,dy\,dt$$

$$+ \int_{t_0}^{t_1} \oint_\Gamma \left[(P-p)\eta + (M-m)\frac{\partial \eta}{\partial n}\right] ds\,dt = 0$$

where →

$$P = \frac{\partial}{\partial n}\nabla^2 u + (1-\mu)\frac{\partial}{\partial s}\underbrace{\left[u_{xx}x_n x_s + u_{xy}(x_n y_s + x_s y_n) + u_{yy}y_n y_s\right]}_{}$$

Bending moment

$$M = -\Big[\underbrace{\mu\nabla^2 u}_{\substack{\text{shearing force}\\ S}} + \underbrace{(1-\mu)\left(u_{xx}x_n^2 + 2u_{xy}x_n y_n + u_{yy}y_n^2\right)}_{\substack{\text{Twisting force}\\ \frac{\partial T}{\partial s}}}\Big]$$

$$\Rightarrow \begin{cases} \rho u_{tt} + \nabla^4 u + f = 0 & \text{in } D \\ \left.\begin{aligned} P - p &= 0 \\ M - m &= 0 \end{aligned}\right\} & \text{on } \Gamma \quad \text{(natural B.C.)} \end{cases}$$

<u>Case I</u>: Clamped plate $\Rightarrow u = \frac{\partial u}{\partial n} = 0$ on Γ

$\therefore \eta = \frac{\partial \eta}{\partial n} = 0$ on Γ

$\therefore$ have to solve equ. of motion in D,

$u = \frac{\partial u}{\partial n} = 0$ on Γ

<u>Case II</u>: Simply supported plate: $u = 0$ on $\Gamma \Rightarrow \eta = 0$ on Γ

$\therefore$ Equ. of motion in D

$u = 0$ on Γ

$M = m$ on Γ

<u>Case III</u> Free boundary (no external moments or bdy. forces)

$$\rho u_{tt} + \nabla^4 u + f = 0 \quad \text{in } D$$

$$\left.\begin{aligned} M &= 0 \\ P &= 0 \end{aligned}\right\} \text{on } \Gamma$$

<u>NOTE</u>: $M = 0$ on Γ,
$\Rightarrow S + \frac{\partial}{\partial s}T = 0$ on Γ.
This B.C. was a major advance when it was discovered - problem was thought overdetermined before!

<u>Plan</u>: start with equation of motion and look for proper BC through Calculus of Variations

<u>Example</u>:
$$(pu')' + \lambda p u - F = 0$$
$$\int \left[(pu')' + \lambda p u - F \right] \eta \, dx = 0$$

$\int F \eta \, dx$ comes from $\delta \int F u \, dx$

$\int \lambda p u \eta \, dx$ comes from $\delta \int \frac{1}{2} \lambda p u^2 dx$

and $\int (pu')' \eta \, dx = pu'\eta \Big|_0^L - \underbrace{\int pu'\eta' dx}_{\text{comes from } \delta \int \frac{1}{2} pu'^2 dx}$

Thus, $\delta \int_0^L \left[-\frac{1}{2} pu'^2 + \frac{1}{2}\lambda p u^2 - Fu \right] dx + \left(pu'\eta \right|_0^L = 0$

i.e. the original equation is the Euler equation for
$$\delta \int_0^L \left(-\frac{1}{2} pu'^2 + \frac{1}{2} \lambda p u^2 - Fu \right) dx = 0$$

and so, proper B.C. are $\quad pu' \delta u \Big|_0^L = 0$

So, either $u = u_0$ at $0, L$ or $pu' = 0$ at ends (both/either?)

e.g. $\alpha_1 u + \alpha_2 u' = 0$

$$\delta \left[\int_0^L \left(-\frac{1}{2} pu'^2 + \frac{1}{2} \lambda p u^2 - Fu \right) dx + \frac{1}{2} k \underbrace{u^2(0)}_{\text{not fixed!}} \right] = 0$$

$$\Rightarrow pu'(0) - k u(0) = 0$$
$$u(L) = 0$$

not fixed!

Also $\delta\left[\int_0^L \left(-\frac{1}{2}pu'^2 + \frac{1}{2}qpu^2 - Fu\right)dx - \alpha p(0)\,u(0)\right] = 0$

$$\Rightarrow \left.\begin{aligned} u'(0) &= \alpha \\ u(L) &= 0 \end{aligned}\right\}$$

Both sets of conditions are perfectly legal; what is natural?

Same trouble for

$$\int_{t_0}^{t} \iint_D \left(\rho u_{tt} + \nabla^4 u - F\right)\eta \, dx\,dy\,dt = 0$$

$\therefore$ Can add any divergence to integral without affecting it and come up with completely different B.C.