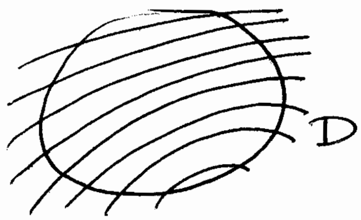
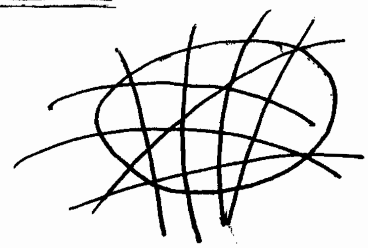
# SUFFICIENT CONDITIONS

## Fields:



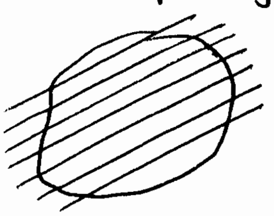

Example



Counter-example

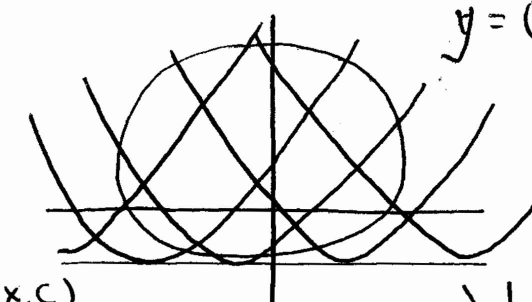
**Defn**

If one and only one curve of the family $y=f(x,c)$ passes through every point of a domain $D$, we say that the family forms a field ("proper field"), Slope $p(x,y)$ at each pt. $(x,y)$ is called slope of the field.

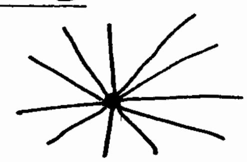


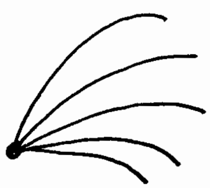
$y = x + c$

$y=(x+a)^2-1$



**Central field**: Assume $y=f(x,c)$ pass through a common point $(x_0,y_0)$.



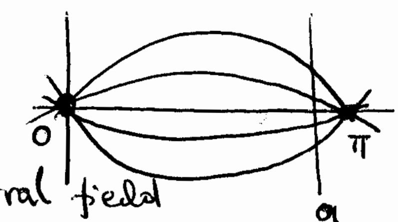


**Example**:

$y = C\sin x$

$0 \le x \le a < \pi$ central field

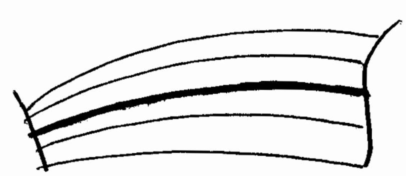
$0 < \delta \le x \le a < \pi$ proper field




Consider

$$I(y)=\int_{x_0}^{x_1} F(x,y,y')\,dx$$

$+ \text{ b.c.}$



We say extremal is successfully imbedded in field if all members of fields are extremals (but do not satisfy BC) and true extremal (i.e. one satisfying BC) is not a "boundary curve (avoid technical definition).

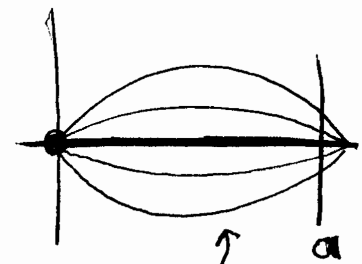
**Example** $I(y) = \int_0^a (y'^2 - y^2)\,dx$

$$y(0) = y(a) = 0, \quad a < \pi$$

Euler eqn: $y'' + y = 0$ ; $y(0) = y(a) = 0 \Rightarrow y(x) \equiv 0$

Consider instead:

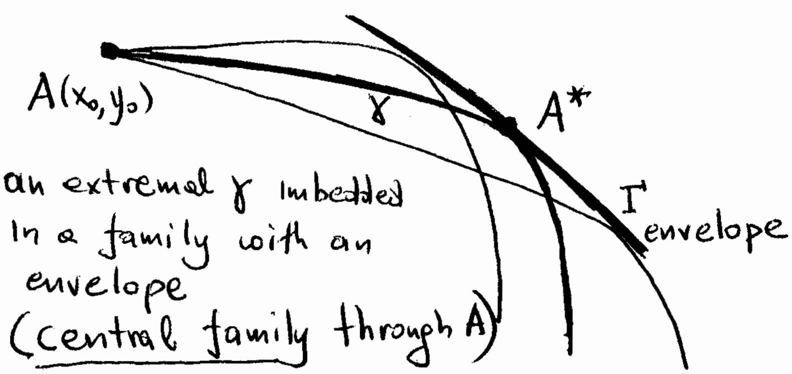
$$\left.\begin{array}{l} y'' + y = 0 \\ y(0) = 0 \end{array}\right\} \Rightarrow y = C \sin x \quad (C = y'(0))$$




gives family of extremals; the real one is successfully imbedded in this central field. (or pencil)

**Definition:** Envelope of a 1-parameter family of curves:

$$\left.\begin{array}{l} f(x,y,C) = 0 \\ f_C(x,y,C) = 0 \end{array}\right\} \quad \text{C-discriminant (see INCE, ODE's)}$$




an extremal $\gamma$ imbedded in a family with an envelope (central family through A)

Fact: If extremal $\gamma$ does not have common points with the envelope $\Gamma$, then extremals of the pencil sufficiently close to $\gamma$ don't intersect.

However: if $\gamma$ does have a pt. $A^*$ in common with $\Gamma$, then curves of the pencil close to $\gamma$ will intersect at points close to pt. $A^*$.

**Defn:** the point $A^*$ is called a point CONJUGATE to A.

Let $y = y(x,c)$ be a pencil of extremals. Take $c$ = slope of $y$ at $A$, i.e. $c = y'(x_0)$ $\quad (y(x_0) = y_0)$.

Envelope is defined by equation:

$$\left.\begin{aligned} y &= y(x,c) \\ \frac{\partial y}{\partial c}(x,c) &= 0 \end{aligned}\right\} \Rightarrow \qquad \text{(solution of Euler's equation)}$$

$$\frac{d}{dx} F_{y'}\left(x, y(x,c), y_x(x,c)\right) - F_y\left(x, y(x,c), y_x(x,c)\right) = 0$$

Take $\dfrac{\partial}{\partial c}$ and set $u \equiv \dfrac{\partial y(x,c)}{\partial c}$

$$\Rightarrow \frac{d}{dx}\left(F_{y'y}\, u + F_{y'y'}\, u'\right) - F_{yy}\, u - F_{yy'}\, u' = 0 \, , \quad u' = \frac{\partial u}{\partial x}$$

or

$$\boxed{\frac{d}{dx}\left(F_{y'y'}\, u'\right) - \left(F_{yy} - \frac{d}{dx} F_{yy'}\right) u = 0} \qquad \left(\begin{matrix}\text{Jacobi} \\ \text{eqn.}\end{matrix}\right)$$

or $$(P u')' - Q u = 0$$

where $P = F_{y'y'}$ , $Q = F_{yy} - \dfrac{d}{dx} F_{yy'}$

The solution $u = \dfrac{\partial y(x,c)}{\partial c}$ vanishes at $A$

$\left( y(x_0,c) = \cancel{0}\, y_0 \Rightarrow u(x_0,c) = 0 \right)$.

If a conjugate point $x_1$ exists in the interval, then we will have $u(x_1,c) = 0$, $u \not\equiv 0$. Clearly, in this case we will have nontrivial solutions to Jacobi's eqn. w. b.c. $u(x_0) = u(x_1) = 0$.

## Necessary conditions: revisited and extended

In considering the second variation, we arrived at the condition (assuming F has 3 ~~(three)~~ derivatives):

$$\frac{1}{2}J = \delta^2 I = \frac{1}{2}\int_{x_0}^{x_1}(Ph'^2 + Qh^2)\,dx \,, \quad h(a)=h(b)=0 \qquad (*)$$

$$P = \cancel{\tfrac{1}{2}} F_{y'y'} \,, \quad Q = \cancel{\tfrac{1}{2}}\left(F_{yy} - \frac{d}{dx}F_{yy'}\right)$$

to arrive at Legendre's necessary condition for the functional

$$I = \int_{x_0}^{x_1} F(x,y,y')\,dx \,, \quad y(x_0) = y(x_1) = 0$$

to have a weak minimum, i.e. $F_{y'y'} \geq 0$. (We showed that Legendre's condition for the second variation to be positive semidefinite, i.e.

$$\delta^2 I \geq 0 \;\rightarrow\; F_{y'y'} \geq 0 \;)$$

To show the converse, we have (cf. Notes p. 22, 22a,b)

**Theorem**: Let $P(x) > 0$, $(x_0 < x < x_1)$; ~~and~~ then the interval $[x_0, x_1]$ contains no points conjugate to $x_0$ if and only if $(*)$ is positive definite for all $h(x)$ w. $h(x_0) = h(x_1) = 0$.

◀ (only if) $\quad J = \displaystyle\int_{x_0}^{x_1} \left\{ Ph'^2 + Qh^2 + \frac{d}{dx}(wh^2) \right\} dx$

((1) pos. def.) ⟸ (no conj. pts.)

$$= \int_{x_0}^{x_1} \left\{ Ph'^2 + 2whh' + (Q+w')h^2 \right\} dx$$

let $w^2 = P(Q+w')^{(*)}$: $\quad = \displaystyle\int_{x_0}^{x_1} P\left(h' + \frac{w}{P}h\right)^2 dx \geq 0 \qquad (2)$

If (2) vanishes for some $h$, then $\quad (P>0)$

(*) Not a given! We must establish that w can be found!

$$\left\{ h' + \frac{w}{P} h \equiv 0 \quad , \quad h(x_0) = 0 \right\} \Rightarrow h(x) \equiv 0.$$

So, $J > 0$. Therefore this part of proof will be complete if we show that the absence of conjugate points guarantees solvability of the ODE

$$w^2 = P(Q + w') \qquad (3)$$

But (3) is a Ricatti equation: let $w = -\frac{u'}{u} P$ to get that $u$ satisfies

$$-\frac{d}{dx}(Pu') + Qu = 0 \qquad (4)$$

which is the Jacobi equation found previously (and also the Euler equ. for $J$). If there are no points conjugate to $x_0$ in $[x_0, x_1]$, then (4) will have a solution that vanishes nowhere in $[x_0, x_1]$ and (3) will be solvable (by continuity, $x_0 - \varepsilon$ will have no conjugate point in $[x_0 - \varepsilon, x_1]$ for $\varepsilon$ sufficiently small, since $u$ depends continuously on I.C.).

To prove the $\underset{(\text{posdef})\ (\text{no conj pb})}{\overset{\text{if}}{(\Longleftarrow)}}$ part of theorem, first prove:

<u>Lemma:</u> If $h$ satisfies (4) with B.C. $h(x_0) = h(x_1) = 0$

then $J[h] = 0$. (obvious since (4) is Euler eq. for $J$)

Now we consider a HOMOTOPY argument: we will connect $J$ to the positive definite functional

$$J_0 = \int_{x_0}^{x_1} h'^2 \, dx \quad (\Rightarrow \underbrace{h'' = 0}_{\text{Euler equ.}})$$

For this functional, if $h(x)\not\equiv 0$ $\overset{(*)}{\phantom{x}}$, $h(x_0)=0$ then $h(x)\neq 0$ for $x\neq x_0$ (indeed $h(x)=\alpha x$, $\alpha\neq 0$), i.e. there is no conjugate point to $x_0$. We define the homotopy family

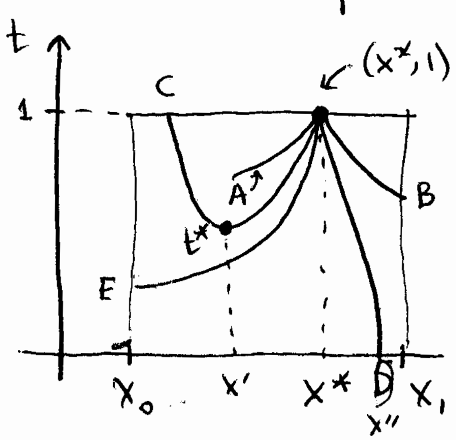
$$H(h;t) = \int_{x_0}^{x_1} \left[t\left(Ph'^2+Qh^2\right)+(1-t)h'^2\right]dx \,, \qquad (5)$$

positive definite $\forall t\in[0,1]$. Then get Euler equ:

$$-\frac{d}{dx}\left\{[tP+(1-t)]h'\right\}+tQh=0 \qquad (6)$$

The solution of (6) with $h(x_0,t)=0$; $h_x(x_0,t)=1$ is continuous in $t$, and $h(x,1)\equiv h(x)$, the solution to (4) satisfying $h(x_0)=0$, $h'(x_0)=1$, while for $t=0$ it becomes $h(x,0)=x-x_0$. If for some $x^*,t_0$ we had $h_x(x^*,t_0)=0$, $h(x^*,t_0)=0$ then $h(x,t_0)\equiv 0$, impossible. So, if $h(x^*,t_0)=0 \Rightarrow h_x(x^*,t_0)\neq 0$.
(second order, linear, homog. ode)

(*) Also, $h(x_1,1)\neq 0$ (otherwise $H(h,1)=0$ by lemma impossible ($h\not\equiv 0$) since $H$ pos.def.). Then, we just need to show no interior point $x^*\in(x_0,x_1)$ is conjugate to $x_0$.




Consider set $(x,t)$: $h(x,t)=0$.
Then $h_x(x,t)\neq 0$ so that (I.F.T.)

($\mathcal{Y}$) $h(x,t)=0 \Longrightarrow x=x(t)$, continuous in neighborhood of solutions to ($\mathcal{Y}$).

Now, assume interior conjugate pt. $(x^*,1)$, and various cases of termination:

(A) impossible by $t$-continuity of $h(x,t)$

(B) if $h(x_1,t_1)=0$ then $(h(x_0,t_1)=0)$, violates pos.def. (lemma)

(C) turning point $\Rightarrow h_x(x',t^*)=0$, $h(x',t^*)=0 \Rightarrow h\equiv 0$, contradiction

(D) would imply $h(x'',0)=0$, impossible

(E) implies $h_x(x_0,t)=0$ for some $t$ ( $h(x,t)=0$ for $x_0$ and some $x=x'''$ arbitrarily close to $x_0$ etc. )

**Corollary:**
~~Note~~: if the functional (2) is non-negative then $x_1$ could possibly be conjugate to $x_0$ since then the use of the lemma is no longer a guarantee ~~that~~ that $h \neq 0$ (i.e. case B could be true)

**Theorem (Jacobi's Necessary Condition)** If $y = y(x)$ is a minimum of the functional $\int_{x_0}^{x_1} F(x, y, y')\,dx$ and $F_{y'y'} > 0$ along $y(x)$, then $(x_0, x_1)$ contains no points conjugate to $x_0$.

◁ We know minimum, $\Rightarrow$ (2) is nonnegative. theorem follows from corollary above ▷

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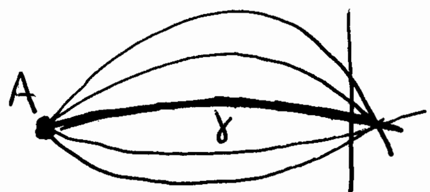
**Corollary:** If Jacobi condition is satisfied on an extremal $\gamma$ (i.e. if there are no conjugate points to A on $\gamma$) then $\gamma$ may be imbeded in a central field, centered at A.




central field

## Hilbert's Invariant Integral:

Consider extremal field with slope function $p(x,y)$: $y' = p(x,y)$. Then

$$\mathcal{H} := \int_P^Q \left\{ \left[ F(x,y,p) - p F_{y'}(x,y,p) \right] dx + F_{y'}(x,y,p)\,dy \right\}$$

is independent of path from P to Q

◁ $\left( \int (P\,dx + Q\,dy) \text{ independent of path if } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \right)$

$$\frac{\partial}{\partial y}\left[F(x,y,p)-pF_{y'}(x,y,p)\right]-\frac{\partial}{\partial x}F_{y'}(x,y,p)$$

$$=F_y+\cancel{F_{y'}p_y}-\cancel{p_yF_{y'}}-pF_{y'y}-pF_{y'y'}p_y-F_{y'x}-F_{y'y'}p_x$$

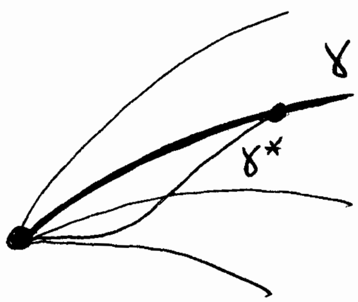
$$=F_y-F_{y'y'}\underbrace{(p_x+pp_y)}_{\frac{dp}{dx}\text{ since }p=y'}-F_{y'y}p-F_{yx}$$

$$=F_y-\frac{d}{dx}F_{y'}=0$$

Now consider extremals $\gamma$ and any other path $\gamma^*$

$$I(\gamma)=\int_\gamma F(x,y,p)\,dx$$

$$=\int_\gamma\Big[F(x,y,p)-p\overbrace{F_{y'}(x,y,p)+F_{y'}(x,y,p)y'}^{=0\text{ since }p=y'}\Big]dx$$




$$=\int_\gamma\Big\{\left[F(x,y,p)-pF_{y'}(x,y,p)\right]dx+F_{y'}(x,y,p)\,dy\Big\}$$

$$=\int_{\gamma^*}\Big\{F(x,y,p)-pF_{y'}(x,y,p)\Big]dx+F_{y'}(x,y,p)\,dy\Big\}$$

NOTE: $p\neq y'$ on $\gamma^*$

$$=\int_{\gamma^*}\left[F(x,y,y')-F(x,y,y')+F(x,y,p)+(y'-p)F_{y'}(x,y,p)\right]dx$$

$$\therefore\ I(\gamma^*)-I(\gamma)=\int_{\gamma^*}\underbrace{\left[F(x,y,y')-F(x,y,p)-(y'-p)F_{y'}(x,y,p)\right]}_{\equiv\ E(x,y,p,y')}dx$$

"Weierstrass E-function"

<u>NOTE</u>: Fix $x$ and $y$ and expand $F(x,y,y')$ in a Taylor series about the point $p=y'$

$$F(x,y,y')=F(x,y,p)+F_{y'}(y'-p)+F_{y'y'}(x,y,\underbrace{q}_{\text{bet. }(y',p)})\frac{(y'-p)^2}{2}$$

$$\therefore\ E(x,y,p,y')=F_{y'y'}(x,y,q)(y'-p)^2/2$$

## Weak ($II_1$) vs. Strong ($II_0$) extrema (Sufficient Conditions)

### For weak extremum:

(i) The curve $\gamma$ is an extremal satisfying B.C.
(ii) The extremal $\gamma$ may be imbedded in field of extremals
(iii) The E-function does not change sign at any point $(x,y)$ close to $\gamma$ and for values of $y'$ close to $p(x,y)$ only.

$$E \geq 0 \Rightarrow \min \quad ; \quad E \leq 0 \Rightarrow \max$$

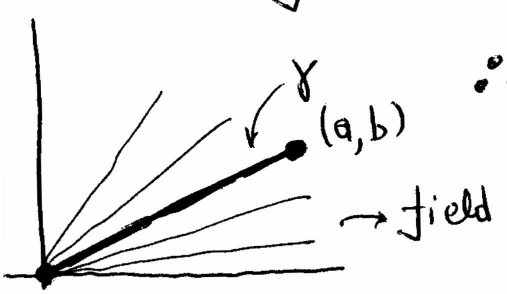
### For strong extremum
(i), (ii) same

(iii) The E-function does not change sign at any point $(x,y)$ close to $\gamma$ for arbitrary values of $y'$;

$$E \geq 0, \min \quad ; \quad E \leq 0, \max.$$

**Example:** $I[y] = \int_0^a y'^3\,dx \quad ; \; y(0)=0, \; y(a)=b. \quad (a,b>0)$

$$y''=0 \Rightarrow y = C_0 x + C_1 \quad \text{extremals}$$




$$\therefore y(x) = \frac{b}{a}x$$

$$E = y'^3 - p^3 - 3p^2(y'-p)$$
$$= (y'-p)^2(y'+2p)$$

Slope of $\gamma$ : $p = \frac{b}{a} > 0$ ; if $y'$ assumes values close to $(b/a)$ then $E \geq 0$

$\therefore$ A weak minimum is achieved on $y = \frac{b}{a}x$

But if $y'$ takes on arbitrary values, then $y'+2p$ may have any sign $\therefore$ E changes sign, no strong minimum.

(88)

$\Rightarrow$ (see G & F p. 149 where it is proved that E having one sign is also a necessary condition).

# Recap

(1) Weak extrema
(i) $\gamma$ extremal
(ii) imbedded in field
(iii) E-function ...

(2) Strong extrema
(i) "
(ii) "
(iii) E-func. keeps sign under strong deviation

$$E(x,y,p,y') = F_{y'y'}(x,y,q)\frac{(y'-p)^2}{2}$$

if $F \in C^3$ wrt $y'$

Replace (iii):

Weak: $F_{y'y'}(x,y,y') \neq 0$ at pts. on $\gamma$

$F_{y'y'} \geq 0$ on $\gamma$ $\Leftrightarrow$ $E \geq 0$
$\phantom{F_{y'y'}} \leq 0$ $\phantom{on}$ $\phantom{\Leftrightarrow}$ $E \leq 0$

Strong extrema: $F_{y'y'}$ doesn't change sign at points $(x,y)$ close to $\gamma$ for arbitrary $y'$.

Ex: $\int_0^a y'^3\,dx$ ; $y(0)=0$, $y(a)=b$ ; $a,b>0$ ; $F_{y'y'} = 6y' = 6\frac{b}{a} > 0$

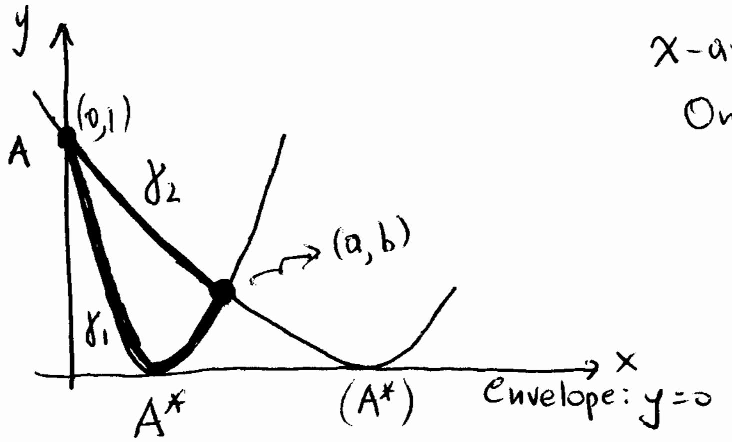
weak minimum

Ex: $I(y) = \int_0^a \frac{y}{y'^2}\,dx$ , $y(0)=1$, $y(a)=b$ , $a>0$, $0<b<1$

Euler: $\frac{d}{dx}F_{y'} - F_y \Rightarrow y'F_{y'} - F = C$

$$\Rightarrow -y'\left(\frac{2y}{y'^3}\right) - \frac{y}{y'^2} = C$$

$$\left.\begin{array}{l} y = (C_1x+C_2)^2 \\ y(0)=1 \end{array}\right\} \Rightarrow \boxed{y = (C_1x+1)^2}$$ , family of parabolas




x-axis is envelope ("C'-discriminant")
On arc AB of $\gamma_1$ there is a conjugate pt. $A^*$ for ($A^* < a$), while on arc $\gamma_2$ there is no conjugate pt ($A^* > a$).
$\therefore$ The Jacobi condition is satisfied on $\gamma_2$ (an extremum can be achieved on this arc).

Must use E function, although $F_{y'y'} = \frac{6y}{y'^4} > 0$ for arbitrary $y'$ (reason:
cannot write $F_y(x,y,y') = F(x,y,p) + (y'-p)F_p + \frac{(y'-p)^2}{2} F_{y'y'}(x,y$
for arbitrary values of $y'$ since function is discontinuous at $y'=0$; rather we can only assert a weak minimum, i.e. use values of $y'$ close to $p$ on $\gamma_2$).

Then: $E(x,y,p,y') = \frac{y}{y'^2} - \frac{y}{p^2} + \frac{2y}{p^3}(y'-p) = \frac{y(y'-p)^2(2y'+p)}{y'^2 p^3}$

$\Rightarrow$ strong minimum is not achieved (E can change sign).

But can use Legendre condition to assert ($F_{y'y'} > 0$ on $\gamma_2$)

$\gamma_2$ provides a weak minimum

Ex: $\min \int_{0}^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,dx$ : $F_{y'y'} = \frac{1}{\sqrt{y}(1+y'^2)^{3/2}} > 0$ for all $y'$

$\Rightarrow$ strong minimum.

Recap

Theorem: $I[y] = \int_a^b F(x,y,y')\,dx$, $y(a)=A$, $y(b)=B$

Suppose that for some admissible $\gamma$
(i) $\gamma$ satisfies Euler eqn.
(ii) Along $\gamma$, $F_{y'y'}>0$ (strengthened Legendre condition)
(iii) The interval $[a,b]$ contains no pts. conjugate to $a$.
(strengthened Jacobi condition).
Then $I(y)$ has weak minimum on $\gamma$

Proof: Recall that

$$\Delta I \equiv I(y+h) - I(y) = \delta I + \delta^2 I + R \,, \quad R \to 0 \text{ as } \|h\| \to 0$$
(weak inf. norm).

$$\delta I = \int_a^b (F_y h + F_{y'} h')\,dx$$

(1) $$\delta^2 I = \frac{1}{2}\int_a^b (F_{yy} h^2 + 2F_{yy'} hh' + F_{y'y'} h'^2)\,dx$$

with $\delta I = 0$

Consider: (2) $\int_a^b [w' h^2 + 2w\, hh']\,dx = 0 = wh^2\Big|_a^b = 0$, $h(a)=h(b)=0$
($w$ arbitrary)

(3) $$\delta^2 I = \frac{1}{2}\int_a^b \left[(F_{yy}+w')h^2 + 2(F_{yy'}+w)hh' + F_{y'y'}h'^2\right]dx$$

Choose $w$ to make integral in (3) perfect square

(4) $$F_{y'y'}(F_{yy}+w') - (F_{yy'}+w)^2 = 0$$

Then (3) becomes:

(5) $$\delta^2 I = \frac{1}{2}\int_a^b \left(h' + \frac{F_{yy'}+w}{F_{y'y'}}\right)^2 F_{y'y'}\,dx$$

For (4) make change of variable $w = -F_{yy'} - F_{y'y'} \frac{u'}{u}$

$$\frac{d}{dx}\left(F_{y'y'} u'\right) - \left(F_{yy} - \frac{d}{dx} F_{yy'}\right) = 0$$

or $\quad (Pu') - Qu = 0 \quad$ (Jacobi eq.)

If $\exists$ solution $u$ of Jacobi eq., $\neq 0$ on $[a, b]$

then $\exists$ sol $\quad w = -F_{yy'} - F_{y'y'} \frac{u'}{u} \in C^1[a, b]$ of (4).

Conds. $\left.\begin{matrix}(ii)\\(iii)\end{matrix}\right\} \Rightarrow \exists$ such $u$ $\quad$ (pp. 104-111 GF).

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(1.1)

Set I Solution

Math. 579, Fall '05

1. Discuss the upper and lower semicontinuity of the arclength functional $I(y) = \int_a^b \sqrt{1+y'^2}\,dx$ with respect to strong variations in the space $C'[a$

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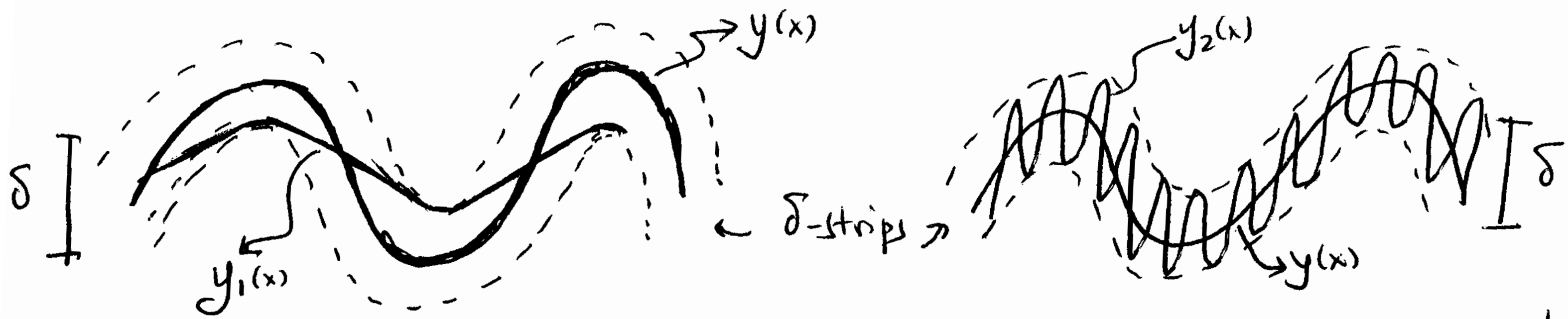



I(y) is lower semicontinuous but not upper semicontinuous in C' with respect to strong variations: i.e. in the norm $\| \ \|_0$, we can have C' functions $y_2(x)$ such that $|y_2(x) - y(x)| < \delta$ but $I[y_2] - I[y] > M \ \forall M > 0$ (we proved this in class by considering the function $y + \varepsilon\cos(x/\varepsilon) \equiv y_2$. But on the other hand, for any $y_1(x)$, $|y_1(x) - y(x)| < \delta$ (i.e. $\|y_1 - y\|_0 < \delta$) ~~then~~ we cannot ~~be~~ get a much smaller arclength than for y: no matter how we "cut corners", in order to "stay on the road" (i.e. within a δ-strip about y(x)) we will end up with an arclength comparable to that of y (More precisely the arclength for $y_1(x)$ cannot be ~~un~~ less than some constant multiple of δ, so ~~ct~~ it → 0 as δ → 0); for any ε > 0 we can find some δ so $-\varepsilon < I[y_1] - I[y]$ if $\|y_1 - y\|_0 < \delta$. The total variation of y needs to be taken into account to turn this into a precise statement.

(1.2)

(2.) Discuss the problem of minimizing $\int_0^\alpha (y'^4 - 6y'^2)\,dx$, $y(0)=0$, $y(\alpha)=\beta$. Consider both $C^1[a,b]$ and $C_p^1[a,b]$ and comment, with reasons, on whether your answers are weak or strong minima.

$$I[y] = \int_0^\alpha (y'^4 - 6y'^2)\,dx \;,\; y(0)=0,\; y(\alpha)=\beta$$

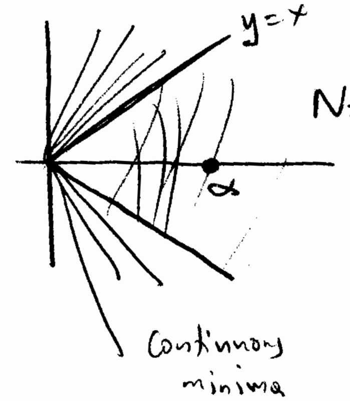
$$F_{y'} = 4y'^3 - 12y' \;,\; F - y'F_{y'} = 6y'^2 - 3y'^4 \;,\; F_{y'y'} = 12(y'^2-1)$$

For relative minimum: $F_{y'y'} \geq 0 \Rightarrow y'^2 \geq 1$.

Euler equation reduces to

$$y' = A \Rightarrow y = Ax + B$$

straight lines




No solution in $C'$

Corner conditions $y'_+ = P,\; y'_- = Q$

$$\left.\begin{array}{ll}(i) & P(P^2-3) = Q(Q^2-3) \\ (ii) & P^2(P^2-2) = Q^2(Q^2-2)\end{array}\right\}$$

which, after some trivial algebraic manipulations, yield the solutions

$$\begin{cases}(P,Q) = (\pm 1, \pm 1) & \text{(continuous lines)} \\ (P,Q) = (\pm\sqrt{3}, \mp\sqrt{3}) & \text{(corners)}\end{cases}$$

Corner solutions satisfy Legendre condition, i.e. correspond to minima.

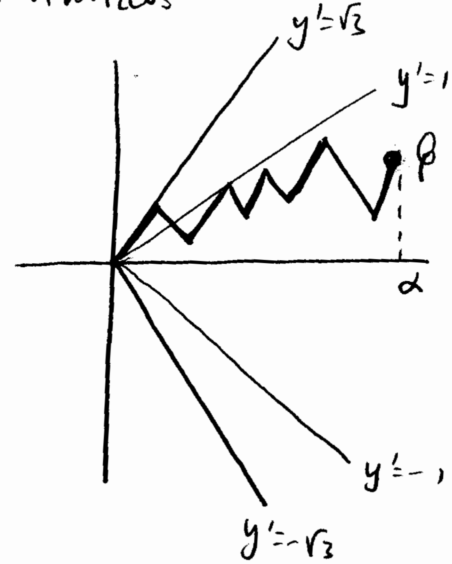
Cases (a) $y \in C'$

(i) $\left|\frac{\beta}{\alpha}\right| < 1$ broken lines inadmissible, straight lines not minima $\Rightarrow$ no solution

(1.3)

(ii) $\left|\frac{\beta}{\alpha}\right| \geq 1$ : straight line $y = \beta/\alpha \, x$ satisfies Legendre condition $\rightarrow$ may be minimizers

(b) $y \in C_p^1$ : (broken lines allowed)




(i) $\left|\frac{\beta}{\alpha}\right| < 1$ : every point can be reached by broken line, slopes $\pm\sqrt{3}$.

They are actually strong minima because $y' = \pm\sqrt{3}$ corresponds to the minimum value of $F(y') = y'^4 - 6y'^2$

(indeed $F_{y'} = 4(y'^2-3)y' = 0$, $F_{y'y'} = 12(y'^2-1)$
i.e. $F_{y'}(\pm\sqrt{3}) = 0$ ; $F_{y'y'}(\pm\sqrt{3}) > 0$ min
while $F_{y'}(0) = 0$, $F_{y'y'}(0) = -12 < 0$ max.

(ii) $1 \leq \left|\frac{\beta}{\alpha}\right| \leq \sqrt{3}$ : now points can be reached by broken lines (strong minima, same reason as before) and straight lines (weak minima, any line can be replaced by arbitrarily close broken lines approximating it in $\| \|_0$ norm, which is a minimum.

(For all points in $\left|\frac{\beta}{\alpha}\right| < \sqrt{3}$ abs. minimum is
$\int_0^\alpha (-9)\, dx = -9\alpha$, attained by (broken) lines w. slopes $\pm\sqrt{3}$

(iii) $\left|\frac{\beta}{\alpha}\right| > \sqrt{3}$ ; no broken lines possible. Now straight lines $y = \beta/\alpha \, x$ are strong minima.

(1.4)

3. Determine whether it is possible to show that every equation $y'' = f(x,y,y')$ is the Euler equation for some functional $\int F(x,y,y')\,dx$. How do we determine the function $F(x,y,y')$ from the function $f(x,y,y')$? Determine whether it is possible to find all functionals for which the extremals are straight lines

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$$I[y] = \int_a^b F(x,y,y')\,dx \longrightarrow y'' = f(x,y,y')?$$

$$\downarrow$$

$$\delta I = 0 \Rightarrow y'' = \frac{y' F_{y'y} + F_{y'x} - F_y}{F_{y'y'}} \quad \text{for } F_{y'y'} \neq 0$$

This means that, in the general case, to determine $F$ we have to solve an equation of the form

$$\boxed{f(x,y,y')\, F_{y'y'} + y' F_{y'y} + F_{y'x} - F_y = 0}$$

which is linear with variable coefficients in $F$

For the case $f = 0$: $(y = Ax + B)$ get

$$\boxed{A F_{zy} + F_{zx} - F_y = 0} \quad (z := y')$$

Special case: $F = F(y')$. (we know that $F_{y'y'} \neq 0$)

i.e. ~~F is at least quadratic in y'.~~

Separation of variables (i.e. $F(x,y,z) = a(x)b(y)c(z)$) yields that $F = \sum_i' G_i\, e^{a_i x + b_i y + \frac{b_i}{Ab_i + a_i} z}$ satisfies this equation. Alternatively, looking for $F = F(y,z)$ find

$$F(y,z) = b(y)\, e^{z/A} + \frac{1}{A}\int_0^z e^{\frac{z-t}{A}}\, a(t)\,dt \; ; \; a, b \text{ arbitrary works.}$$

Answer non-unique

More generally, ~~if~~ ~~f~~ get: let $y' = z$ (1.5)

$$\cancel{f F_{z}} \quad f(x,y,z) F_{zz} + z F_{zy} + F_{zx} - f_y = 0 \quad ;$$

$$\frac{\partial}{\partial z} : \quad f_z F_{zz} + f F_{zzz} + \cancel{F_{zy}} + z F_{zzy} + F_{zzx} - \cancel{F_{zy}} = 0$$

Let $G = F_{zz}$:

$$f G_z + z G_y + G_x + f_z G = 0$$

$1^{st}$ order PDE, solvable in terms of arbitrary fns.

If $f = 0$: $\quad G_x + z G_y = 0 \Rightarrow G = \phi(y - zx, z)$

↳ arbitrary

Now $\quad F_{zz} = \phi(y - zx, z) \quad \Rightarrow$

$$\boxed{F = a(x,y) + z\, b(x,y) + \int_0^z (z-t)\, \phi(y - t x, t)\, dt}$$

with $\dfrac{\partial a}{\partial x} = \dfrac{\partial b}{\partial y}$

i.e. $\quad a = \dfrac{\partial \mu}{\partial y}, \; b = \dfrac{\partial \mu}{\partial x} \quad ; \; \mu(x,y)$ arbitrary.

Special case $\quad \phi = \phi(z)$; $\quad F = \displaystyle\int_0^z (z-t)\,\phi(t)\, dt$

$\Rightarrow \phi = F_{zz}(z)$

arbitrary fn. of $z$.

4. (i) $\int_{x_0}^{x_1} (ay'^2 + byy' + cy^2)\,dx$ ; $y(x_0) = y_0$, $y(x_1) = y_1$ , $a \neq 0$ (1.6)

$$\left.\begin{aligned} F_{y'} &= 2ay' + by \\ F_{y'y'} &= 2a \neq 0 \end{aligned}\right\} \quad F - y'F_{y'} = cy^2 - ay'^2$$

$y'_+ = P$, $y'_- = Q$

<u>Corner conditions</u> : $F_{y'} = 2aP = 2aQ \Rightarrow P = Q$ ; <u>no corners</u>

(ii) $\int_{x_0}^{x_1} y'^3\,dx$ , $y(x_0) = y_0$ , $y(x_1) = y_1$

$$F_{y'} = 3y'^2 \; ; \; F_{y'y'} = 6y' \; ; \; F - y'F_{y'} = -2y'^3$$

Corners:
$$\left.\begin{aligned} F_{y'} &: \; 3P^2 = 3Q^2 \\ F - y'F_{y'} &: \; P^3 = Q^3 \end{aligned}\right\} \Rightarrow P = Q$$ ; <u>no corners.</u>

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5. $\int_0^1 (y''^2 - 2xy)\,dx$ ; $y(0) = y'(0) = 0$, $y(1) = \frac{1}{120}$ ; $\delta y = y + h$

$h(0) = h(1) = h'(0) = 0$

$h(1)$ free

$$\delta I = \int_{x_0}^{x_1} (F_y h + F_{y''} h'')\,dx = 0 \quad \text{for weak extremum}$$

$$= \int_{x_0}^{x_1} F_y h\,dx + h' F_{y''}\Big|_{x_0}^{x_1} - h(F_{y''})'\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} h \frac{d^2}{dx^2}(F_{y''})\,dx = 0$$

$$\Rightarrow \int_{x_0}^{x_1} h\left(F_y + \frac{d^2}{dx^2} F_{y''}\right)dx + h'(1)\, F_{y''}\Big|_{x=x_1} = 0$$

$$\Rightarrow \quad F_y + \frac{d^2}{dx^2} F_{y''} = 0 \; ; \quad y(0) = y'(0) = 0 \; ; \; y(1) = \frac{1}{120}$$

$y(1)$ unspecified

(we could let $h'(1)$ be free $\Rightarrow F_{y''}\big|_{x=x_1} = 0$ ,
~~β~~ natural BC → but we learned that later!)

$$\Rightarrow \left.\begin{aligned} F &= y''^2 - 2xy \\ F_y &= -2x \\ F_{y''} &= 2y'' \\ &\Downarrow \\ \frac{d^2}{dx^2} F_{y''} &= 2y^{(iv)} \end{aligned}\right\}$$

Euler

$y^{(iv)} = x \Rightarrow$

$y = \frac{1}{120} x^5 + a x^2 (x-1)$

with $a$ free satisfies 3 BC

(i) If we demand $F_{y''}\Big|_{x=x_1 = 1} = 2y'' = \frac{1}{6} x^3 + 6ax - 2a \Big|_{x=1} = 0$

$\Rightarrow \frac{1}{6} + 6a - 2a = 0 \Rightarrow a = -\frac{1}{24}$

(ii) Instead, we can find this by brute force, by minimizing the functional over $a$:

$I(y) = I\left[ x^2 \left( \frac{1}{120} x^3 + a(x-1) \right) \right] =$

$y'' = \frac{1}{6} x^3 + 6ax - 2a$

$(y'')^2 = \frac{1}{36} x^6 + 2a x^4 - \frac{2}{3} a x^3 + 36 a^2 x^2 - 24 a^2 x + 4a^2$

$y''^2 - 2xy = \frac{1}{90} x^6 + \frac{4}{3} a x^3 + 36 a^2 x^2 - 24 a^2 x + 4a^2$

$\int_0^1 (y''^2 - 2xy)\, dx = \frac{1}{630} + \frac{1}{3} a + 4a^2 = \phi(a)$

$\frac{d\phi}{da} = 0 \Rightarrow \frac{1}{3} + 8a = 0 \Rightarrow a = -\frac{1}{24}$

i.e. $y_0(x) = \frac{x^2}{120} (x^3 - 5x + 5)$, $\boxed{I(y_0) = -.005}$

minimum