Quadratic functionals

(Mikhlin) "The problem of the minimum of a quadratic functional" Holden-Day, 1965

Naturally connected with Linear PDE.

(1) ~~1/variable~~: By its nature, problem poses a "paradox":

Euler equation require more differentiability than functionals.

eg:

$$\Delta u = -f(x,y) \longleftrightarrow I(u) = \iint_\Omega \left(|\nabla u|^2 - 2f\right) dx\,dy$$

$$u|_{\partial\Omega} = 0$$

PDE requires C^2, v.p. only $C_p^{(1)}$ in $\Omega \cup \partial\Omega$. Is it true that extremum is C^2 even if class over which we minimize is only C'? In 1d this is ensured by the thm. of Du Bois-Reymond. Not so easy to show this in $n \geq 2$ dims. Moreover, manner in which BC are satisfied must be clarified.

$d(x_n, x_m) \to 0$, $n,m \to \infty$ → Cauchy sequences converge.

Hilbert space: complete inner product space

Linear space: ~~inner prod~~

Banach space: complete normed linear space

(every metric space has unique completion)

Inner product

$(x+y, z) = (x,z) + (y,z)$

$(\alpha x, y) = \alpha(x,y)$

$(x,y) = \overline{(y,x)}$

$(x,x) > 0 \quad, x \neq 0$

Norm $\|x\|^2 = (x,x)$

Bilinear functional: $\Phi(u,v)$

$\Rightarrow \Phi(u,u)$: homogeneous quadratic functional $\to \Phi(u)$

$$\triangleleft\ \Phi(au+bv) = |a|^2\Phi(u) + a\bar{b}\,\Phi(u,v) + \bar{a}b\,\Phi(v,u) + |b|^2\Phi(v)$$

Quadratic functional

$$F(u) = \underset{\text{HQ}}{\Phi(u)} + l_1 u + \overline{l_2 u} + C$$

l_1, l_2 → linear fcts.

(14.2)

Operators in a Hilbert space (with dense domain)

A positive : $(Au, u) > 0$, $(u \neq 0)$

A pos. def. : $(Au, u) \geq \gamma^2 \|u\|^2$ $(\gamma > 0)$

$(\lambda)^2$ $Ax_i = \lambda_i x_i \Rightarrow x_1 \perp x_2$

$\star A$ symmetric: (Au, u) (~~contained in its adjoint~~) $(Au, v) = (u, Av)$ $u, v \in D_A$

A self adjoint : $(Au, v) = (u, Av)$ $D_A = D_{A^*}$, $Ax = A^* x$ $x \in D_A$

Lemma: A positive $\Rightarrow$ $Au = f$ has at most one solution

◁ if u_1, u_2: $A(\underbrace{u_1 - u_2}_{u}) = 0$ ▷

i.e. $(Au, u) = 0$

Theorem 1 : A positive. If

$$Au = f \qquad (1)$$

has solution, then

$$F(u) = (Au, u) - (u, f) - (f, u) \qquad (2)$$

assumes minimum at u; conversely, minimizer satisfies (1)

◁

[A : $D_A \to H$ symmetric (~~Dense domain~~) Linear

$\Updownarrow$

$D_A \subset D_{A^*}$, $Ax = A^* x$ $\forall x \in D_A$

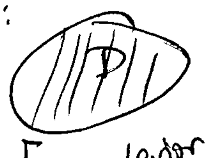
| D_A dense, A self adjoint: iff A symmetric, $D_A = D_{A^*}$]

Eigenvalue Problems etc.

$Au = f$; A selfadjoint, pos. def.

$$(Au, v) = (u, Av)$$
$$(Au, u) > 0 \quad \text{for all } u \neq 0$$

Ex:



Γ exterior

D

$$\Delta u = 0$$
$$u = 1 \text{ on } \Gamma$$
$$u = O\left(\tfrac{1}{r}\right) \text{ as } r \to \infty$$

$$C(u) = \frac{1}{4\pi} \int_D |\operatorname{grad} u|^2 \, dx$$

Ex:

interior

$$\Delta u = -2, \quad u = 0 \text{ on } \Gamma, \qquad S(u) = \int \left(|\nabla u|^2 - 2u \right) dx$$

Ex

$$\Delta u + \lambda u = 0 \quad \text{in } D, \qquad u = 0 \text{ on } \Gamma$$

$$\lambda = \min \frac{(Ax, x)}{(x, x)}$$

Ex: $Lu + f(u) = 0$ in D, $Bu = 0$ on Γ

e.g. $\left.\begin{array}{l} \Delta u + e^u = 0 \\ u = 0 \end{array}\right\} \dots$

Suppose we have solution $u_0(x)$; $U = U(x,t)$

Let $U_t = LU + f(U)$; $BU = 0$ for all t

$U = u_0(x) + \varepsilon V(x,t)$ $\leftrightarrow$ stability

$$\varepsilon V_t = Lu_0 + \varepsilon LV + f(u_0) + f'(u_0)(U - u_0)$$

$$\varepsilon V_t = \varepsilon LV + f'(u_0)\cancel{\varepsilon} V$$

$$V_t = LV + f'(u_0) V$$

$$v_t = Lv + f'(u_0)\, v$$

Then $\quad v = \sum \alpha_n e^{-\sigma_n t}\, \omega_n(\underline{x})$

$$\left. \begin{aligned} L\omega + [\sigma + f'(u_0)]\,\omega &= 0 \\ B\omega &= 0 \end{aligned} \right\}$$

all σ_n must be >0, otherwise a term would grow (in fact $\mathrm{Re}\,\sigma_n \geq 0$), perturbation would take off.

$Au = f$: $\exists$ unique solution or $\left. \begin{aligned} Au_1 &= f \\ Au_2 &= f \end{aligned} \right\} \Rightarrow A(u_1 - u_2) = 0$

Pos def: $A\phi = 0 \Rightarrow (A\phi, \phi) = 0 \Rightarrow \phi \equiv 0$

Interested in: $\alpha = (f, u)$

$$\alpha = (f, u) = (Au, u) = (u, Au) = (u, f)$$

$$\therefore \alpha = (f, u) + (u, f) - (Au, u) \qquad \left((f,u) + (u,f) = 2\,\mathrm{Re}\,(f,u) \right)$$

Maximum thm: Let $F(v) = (f, v) + (v, f) - (Av, v)$

Then if $Au = f$ has a solution u, this u is the one and only one admissible fn. which maximizes $F(v)$.
Conversely, if in the class of admissible functions there exists one which maximizes $F(v)$, then it is the solution of $Au = f$.

(a) Let u be solution of $Au = f$. Must show that $F(v) < F(u)$ for all $v \neq u$. Let $u - v = h$. Then

$$F(u)-F(v)=(f,u)+(u,f)-(Au,u)-(f,v)-(v,f)+(Av,v)$$
$$=(f,h)+(h,f)-(Au,h)-(Av,h)+(A\xi h,h)$$
$$=(f-Au,h)+(h,f-Au)+(Ah,h)$$
$$=(Ah,h)>0 \text{ since } h\neq 0 \Rightarrow F(u)>F(v)$$

(b) Let u maximize F. Must show that u is solution of $Au=f$.

(u maximizes F) $\Rightarrow F(u)\geq F(u+\varepsilon\eta)$

$\Rightarrow$ with η fixed the r.h.s. as a function of ε has a max. at $\varepsilon=0$

$$\frac{d}{d\varepsilon}F(u+\varepsilon\eta)\Big|_{\varepsilon=0}=0$$

or $$(f-Au,\eta)+(\eta,f-Au)=0$$

True for any η, i.e. also for $i\eta$:

$$(f-Au,\eta)-(\eta,f-Au)=0$$

$$\Rightarrow (f-Au,\eta)=0 \quad \forall\eta$$

$\alpha=\max_{v\in S}F(v)$; for real valued:

$$F(v)=2(f,v)-(Av,v)$$

Mikhlin, "Variational Methods in Mathematical Physics"

Ex: $-u'' = f(x)$; $V = \int_0^\ell \left[\frac{1}{2}u'^2 - f(x)u\right] dx$

$u(0) = u(\ell) = 0$

$$V(u+h) - V(u) = \int_0^\ell u'h' + \frac{1}{2}\int_0^\ell h'^2 - \int_0^\ell fh$$

$$= u'h - \int_0^\ell \underset{-f}{\underset{\parallel}{u''}} h + \frac{1}{2}\int_0^\ell h'^2 - \int_0^\ell fh$$

$$= \frac{1}{2}\int_0^\ell h'^2 > 0$$

Let $Au = -u''$, $u(0) = u(\ell) = 0$

$$F(v) = (f,v) + (v,f) - (Av,v)$$

$$= 2(f,v) - (Av,v) \qquad \int v''v = v'v\Big| - \int v'^2$$

$$= 2\int_0^\ell fv - \int_0^\ell v'^2$$

Solution maximizes $\left[2\int_0^\ell fv - \int_0^\ell v'^2\right]$

(or minimizes : $\left[\frac{1}{2}\int v'^2 - \int_0^\ell fv\right]$)

Theorem: Let $R(v) = \dfrac{|(f,v)|^2}{(Av,v)}$, $v \neq 0$

$(R(v) = R(cv)$, any $c \neq 0)$

Then $F(v)$ and $R(v)$ have same maximum.

Since $R(u) = \dfrac{|(f,u)|^2}{(Au,u)} = \dfrac{(Au,u)^2}{(Au,u)} = (f,u) = \alpha$

$(\because Au = f)$

It is clear that $\max R \geq \max F$ $\left(\begin{array}{l}\text{because } F(u) = \alpha \\ \text{at its max.}\end{array}\right)$

Let z be a function which yields the maximum value for R. Then choose

$$C = \frac{(f,z)}{(Az,z)}.$$

(u maximizes $F(u) \Rightarrow F(u+\varepsilon\eta) \le F(u)$

$\therefore F(u+\varepsilon\eta)$ has max for $\varepsilon = 0 \Rightarrow \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F(u+\varepsilon\eta) = 0$

$$\Rightarrow 0 = \int (2u'\eta' + 4\eta) + 2u(1)\,\eta(1)$$

$$0 = \int \underbrace{(2u''+4)}_{0}\,\eta\,dx + \underbrace{2u'(0)}_{0}\eta(0) + 2\underbrace{(u'(1)+u(1))}_{0}\eta(1) \Bigg)$$

* LEAST SQUARES

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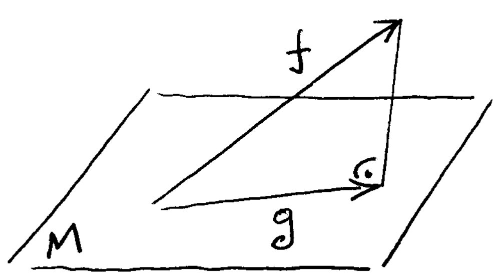
Let f be arbitrary element in complex Hilbert space

M = n-dim. closed linear manifold in H

Look for an element $g \in M$ which is closest to f, i.e. for which $\|f-g\|$ is minimized.



Def: g is called the projection of f on M & is denoted by Pf.

Introduce a basis $\{\Phi_i\} \in M$

$$g = \sum_{i=1}^{n} a_i \Phi_i(x)$$

$$\|f-g\|^2 = \|f - \Sigma a_i \Phi_i\|^2$$

$$= \|f\|^2 + \|\Sigma a_i \Phi_i\|^2 - \cancel{2}\,\Sigma a_i (f,\Phi_i) - \Sigma \bar{a}_i \overline{(f,\Phi_i)}$$

$$= \|f\|^2 + \sum^{n} |(f,\Phi_i) - a_i|^2 - \sum^{n} |(f,\Phi_i)|^2$$

To minimize, take $a_i = (f, \Phi_i)$

(1) $$g = Pf = \sum_{i=1}^{n} (f,\Phi_i)\,\Phi_i$$

— · —

Then

$$F(cz) = \bar{c}(f,z) + c(z,f) - c\bar{c}(Az,z)$$
$$= \frac{|(f,z)|^2}{(Az,z)} = R(z)$$

$\therefore \quad \max F \geq \max R$

$$\Rightarrow \alpha = \max_v F(v) = \max_v R(v) \qquad (\underset{\text{maximum}}{v \to cv})$$

Ex: $\underset{u(0)=u(1)=0}{-u'' = 2}$ $\qquad u = x(1-x)$

Take $\quad Au = -u''$, $u(0)=u(1)=0$

$\therefore \quad u = x(1-x)$ maximizes $F(v)$ $\quad (\min \; -F(v))$

$$\therefore \min_{v\in S}\left[-F(v)\right] = \min_{v\in S}\left[(Av,v) - 2(f,v)\right]$$
$$= \min_{v\in S}\int_0^1 (-vv'' - 4v)\,dx$$
$$= -F(\underset{x(1-x)}{\underset{\parallel}{u}}) = -\frac{1}{3}$$

Consider $v = x(2-x)$; then $-F(v) = -\frac{4}{3} < -\frac{1}{3}$

but v does not satisfy BC (not admissible).

or: $\quad -u'' = 2$ $\qquad$ Exact soln. $u = 3 - x^2$

$\quad u'(0) = 0$ $\qquad$ minimize $(-F(u))$

$\quad u'(1) + u(1) = 0$

$$-F(u) = \int_0^1 (-uu'' - 4u)\,dx$$

$$-F(v) = \int_0^1 (-vv'' - 4v)\,dx = \int_0^1 (v'^2 - 4v)\,dx - vv'\Big|_0^1$$
$$= \int_0^1 (v'^2 - 4v)\,dx + v^2(1)$$

$\therefore \quad u = 3 - x^2$ minimizes $\left[\int_0^1 (v'^2 - 4v)\,dx + v^2(1)\right]$

Let $v_1, \dots, v_n$ be an arbitrary basis in M (not O.N.)

$$Pf = \sum \alpha_i v_i(x)$$

$(f - Pf, v_i) = 0$ for all $i \Rightarrow (f, v_i) = (Pf, v_i)$ $\forall i$

$$Pf = \sum \alpha_i v_i \Rightarrow (Pf, v_j) = \sum_i \alpha_i (v_i, v_j)$$

$$\overset{\parallel}{(f, v_j)} = \sum_i \alpha_i (v_i, v_j)$$

$$F = BA \Rightarrow A = B^{-1} F$$

$$F = \begin{pmatrix} (f, v_1) \\ \vdots \\ (f, v_n) \end{pmatrix}, \quad A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}; \quad B = ((v_i, v_j))$$

$$\therefore Pf = \sum c_{ij} (f, v_i) v_j \; ; \quad c_{ij} = B^{-1} \qquad (2)$$

NOTE: P is self-adjoint, $(Pf, g) = (f, Pg) = (f, g)$

(i) $(Pf, g) = \sum_{i=1}^{n} (f, \Phi_i)(\Phi_i, g)$ from (1)

$(f, Pg) = \sum_{i=1}^{n} (f, \Phi_i) \overline{(g, \Phi_i)} = (Pf, g)$

(ii) P non-singular: $(Pf, f) = \sum (f, \Phi_i)(\Phi_i, f) = \sum |(f, \Phi_i)|^2 \geq 0$

(iii) $P^2 = P$.

$Au = f$, (S.A, P.D., Lin) $u \in \mathcal{D}$, dense submanifold of Hilbert sp. $\mathcal{H}$.

Max principle: $\max\limits_{v \in \mathcal{D}} F(v) = F(u)$

where $F(v) = (f, v) + (v, f) - (Av, v)$

$$\max_{v \in E_n} F(v) \leq \max_{v \in \mathcal{D}} F(v) = F(u)$$

Define P_n as the projector operator on E_n

(Can use (1) or (2) to get explicit representation of $P_n \zeta$ for $\zeta \in \mathcal{D}$.)

for $v \in E_n$ we have

(3) $$F(v) = (f, v) + (v, f) - (Av, v)$$
$$= (f, P_n v) + (P_n v, f) - (Av, P_n v)$$
$$= (P_n f, v) + (v, P_n f) - (P_n A v, v)$$

The functional
$$F(v) = (P_n f, v) + (v, P_n f) + (P_n A v, v)$$

is just the maximizing functional associated with the n-dimensional equation:

(4) $$P_n A u_n = P_n f, \quad u_n \in E_n$$

Note: $P_n A$ is linear, self adjoint, pos. def

$(P_n A z, w) = (Az, P_n w) = (Az, w) = (z, Aw) = (P_n z, Aw) = (z, P_n A w)$

$(P_n A z, z) = (Az, P_n z) = (Az, z) \geqslant 0$, $z \neq 0$.

Among all functions $\in E_n$, the unique solution of (4), u_n, maximizes $F_n(v)$, that is

$$\max_{v \in E_n} \left\{ (P_n f, v) + (v, P_n f) - (P_n A v, v) \right\} = F(u_n) \qquad (5)$$

Rayleigh-Ritz procedure amounts to finding function $\in E_n$ which maximizes the lhs of (5); equivalent to solving (4).

Let $v_1, \dots, v_n$ basis in E_n. Then for $u_n \in E_n$

$$u_n = \sum_{k=1}^{n} a_k v_k \qquad (6)$$

Must find a_k: $\quad P_n A u_n = P_n f \quad$ (approximate solution)

$$(P_n A u_n, v_j) = (P_n f, v_j) \quad \Rightarrow (A u_n, P_n v_j) = (f, P_n v_j)$$

$$(A u_n, v_j) = (f, v_j)$$

$$(A \sum a_k v_k, v_j) = (f, v_j)$$

$$\sum a_k (A v_k, v_j) = (f, v_j), \quad j = 1, \dots, n \qquad (7)$$

$$\sum a_k (A v_j, v_k) = (f, v_j) \qquad \text{(Grammian)}$$

$$u_n = \sum_{j=1}^{n} a_j v_j$$

with $\quad F(u_n) = (f, u_n) + (u_n, f) - (A u_n, u_n)$

Substitute $u_n = \sum a_k v_k$ to obtain approximation to $F(u_n)$:

(8) Claim: $F(u_n) = (f, u_n)$ $\quad \left((f, u_n)_P = (Au_n, u_n)_P \right)$ $\quad \underline{P_n A u_n = P_n f}$

$\triangle$ $(P_n f, u_n) = (f, P_n u_n) = (f, u_n)$; $u_n \in E_n$

Solving $Au = f$, A real:

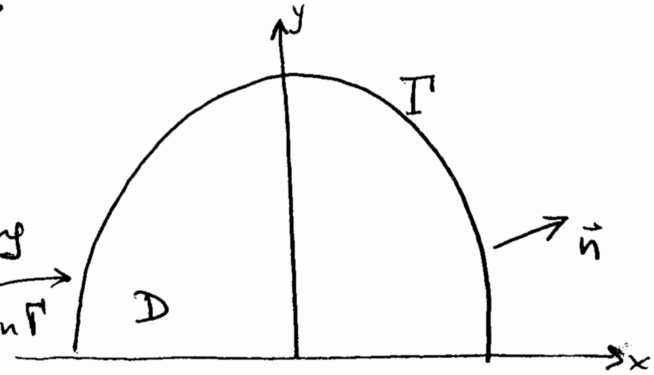
(9) $\quad F(u) = (Au, u) - 2(u, f)$

(10) $\quad u_n = \sum_1^n a_k v_k(x)$

$$F(u_n) = \left(A \sum a_j v_j, \sum a_k v_k \right) - 2\left(\sum a_k v_k, f \right)$$

$$\left(\sum a_j A v_j, \sum a_k v_k \right) - 2\left(\sum a_k v_k, f \right)$$

(11) $\Rightarrow \sum_{j,k} (Av_j, v_k) a_j a_k - 2 \left\{ \sum (v_j, f) a_j \right.$



Example (Mikhlin, p.p. 421-425)

(12) $\nabla^4 w = \frac{q}{\beta}$ in $\mathcal{D}$; $q(x,y)$ = bending

(13) $\quad \beta = \frac{Eh^3}{12(1-\sigma^2)}$, $w' = 0$ on Γ

Natural BC: $\rightarrow$ $\nabla^2 w - \left(\frac{1-\sigma}{\rho} \right) \frac{\partial w}{\partial n} + k \frac{\partial w}{\partial n} = 0$ on Γ (14)

$$F(w) = \iint_D \left[(w_{xx} + w_{yy}) + 2(1-\sigma)(w_{xy}^2 - 2 w_{xx} w_{yy}) \right] dx\,dy$$

$$+ k \int_\Gamma \left(\frac{\partial w}{\partial n} \right)^2 ds - 2 \iint_{\mathcal{D}} \frac{q}{\beta}\, dx\,dy$$

$\sigma = 0$, $k = 1$; $\frac{q}{\beta} = 1$ (not necessary, just convenient)

Consider $x^{2k} y^{l} (1 - x^2 - y^2)$; $k = 0, 1, \dots$; $l = 1, 2, \dots$

$$W_6 = \sum_{k=1}^{6} a_k v_k(x) \; ; \quad v_1 = y(1-x^2-y^2) \; , \quad v_2 = y^2(1-x^2-y^2)$$

$$v_3 = x^2y(1-x^2-y^2) \; , \quad v_4 = y^3(1-x^2-y^2)$$

$$v_5 = x^2y^2(1-x^2-y^2) \; , \quad v_6 = y^4(1-x^2-y^2)$$

$$(Av_i, v_j) = -\iint_D (\nabla^4 v_i) v_j \, dx dy$$

$$= \iint_D (\nabla^2 v_j \, \nabla^2 v_i + 2\, \partial_{xy} v_i \, \partial_{xy} v_j - \partial_x^2 v_i \, \partial_y^2 v_j) \, dx dy$$

Equ. (7) becomes

$$26.1994\, a_1 + 21.3333\, a_2 + \cdots + 17.6667\, a_6 = .2667 \qquad \left(\begin{array}{c}\text{P. 422}\\ \text{Mikhlin}\end{array}\right)$$

Find: $a_1 = .02217, \; \ldots, \; a_6 = -.01083$ etc. (all comparable size – not a good place to truncate!)

System (7) can be considerably simplified if

(15) $\qquad (Av_k, v_j) = \delta_{kj}$

Then (7) reduces to

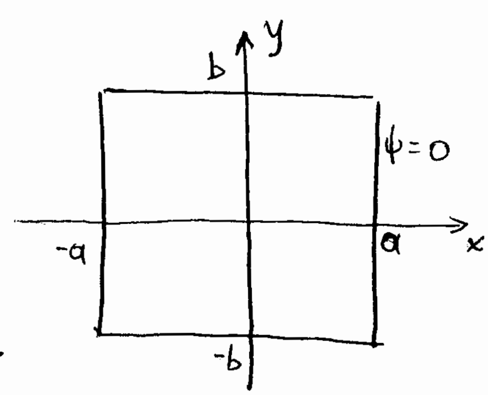
(16) $\qquad a_j = (f, v_j), \; j = 1, \ldots, n$

Thus

(17) $\qquad \underline{u_n = \sum_{j=1}^{n} (f, v_j) v_j}$

Example:

$$-\Delta\psi = 1 \text{ in } \square$$
$$\psi = 0 \text{ on } \partial\square$$



Separation of variables:

$$\psi(x,y) = \frac{16 a^2 b^2}{\pi^4} \sum_{m,n=1,3,5}' \frac{\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{mn(b^2m^2 + a^2n^2)}$$

Set $\quad u_{mn} = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

$$(-\Delta u_{mn}, u_{pq}) = 0 \quad \text{if } m \neq p \text{ or } n \neq q.$$

$$\psi_1(x,y) = \frac{16a^2b^2}{\pi^2(a^2+b^2)}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$\psi_2(x,y) = \frac{16a^2b^2}{\pi^2(a^2+b^2)}\left[\frac{\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}}{a^2+b^2} + \frac{\sin\frac{\pi x}{a}\sin\frac{3\pi y}{b}}{3(9a^2+b^2)} + \frac{\sin\frac{3\pi x}{a}\sin\frac{\pi y}{b}}{9(a^2+3b^2)}\right]$$

$$\psi_3(x,y) = \ldots$$

for R-R, could take u_{mn} – not a good choice for numerical work however

Now $(x^2-a^2)(y^2-b^2)(a_1+a_2x^2+\ldots)$ satisfies BC

Take
$$v_1 = (x^2-a^2)(y^2-b^2)$$
$$v_2 = v_1 x^2$$
$$v_3 = v_1 y^2$$
etc.

(Bad for Physics – good numerically)

$$(Av_i, v_k) = -\int_{-b}^{b}\int_{-a}^{a}(\Delta v_i)\, v_k \, dx\, dy$$

$$(\quad)a_1 + (\quad)a_2 + (\quad)a_3 + \ldots = (\quad)$$

(details in Mikhlin)