Lecture 3
Convex sets

- subspace, affine set, convex set, convex cone
- simple examples and properties
- combination and hulls
- ellipsoids, polyhedra, norm balls
- affine and projective transformations
- separating hyperplanes
- generalized inequalities

Subspaces

$S \subseteq \mathbf{R}^n$ is a *subspace* if

$$x, y \in S, \quad \lambda, \mu \in \mathbf{R} \quad \Longrightarrow \lambda x + \mu y \in S$$

geometrically: $x, y \in S \Rightarrow$ plane through $0, x, y \subseteq S$

representations

$$\begin{aligned} \mathbf{Range}(A) &= \{Aw \mid w \in \mathbf{R}^q\} \\ &= \{w_1 a_1 + \cdots + w_q a_q \mid w_i \in \mathbf{R}\} \\ &= \mathbf{Span}(a_1, a_2, \ldots, a_q) \end{aligned}$$

where $A = \left[\, a_1 \;\; \cdots \;\; a_q \,\right]$

$$\begin{aligned} \mathbf{Nullspace}(B) &= \{x \mid Bx = 0\} \\ &= \{x \mid b_1^T x = 0, \ldots, b_p^T x = 0\} \end{aligned}$$

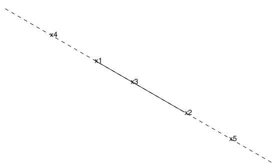
where $B = \begin{bmatrix} b_1^T \\ \vdots \\ b_p^T \end{bmatrix}$

Affine sets

$S \subseteq \mathbf{R}^n$ is *affine* if

$$x, y \in S, \quad \lambda, \mu \in \mathbf{R}, \quad \lambda + \mu = 1 \Longrightarrow \lambda x + \mu y \in S$$

geometrically: $x, y \in S \Rightarrow$ line through $x, y \subseteq S$



representations

range of affine function

$$S = \{Az + b \mid z \in \mathbf{R}^q\}$$

via linear equalities

$$\begin{aligned} S &= \{x \mid b_1^T x = d_1, \ldots, b_p^T x = d_p\} \\ &= \{x \mid Bx = d\} \end{aligned}$$

Convex sets

$S \subseteq \mathbf{R}^n$ is a **convex set** if

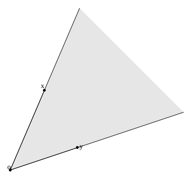
$$x, y \in S, \quad \lambda, \mu \geq 0, \quad \lambda + \mu = 1 \Longrightarrow \lambda x + \mu y \in S$$

geometrically: $x, y \in S \Rightarrow$ segment $[x, y] \subseteq S$ . . . many representations

$S \subseteq \mathbf{R}^n$ is a **convex cone** if

$$x, y \in S, \quad \lambda, \mu \geq 0, \quad \Longrightarrow \lambda x + \mu y \in S$$

geometrically: $x, y \in S \Rightarrow$ 'pie slice' between $x, y \subseteq S$



. . . many representations

Hyperplanes and halfspaces

hyperplane: $\{x \mid a^T x = b\}$ $(a \neq 0)$

affine; subspace if $b = 0$

useful representation: $\{x \mid a^T(x - x_0) = 0\}$

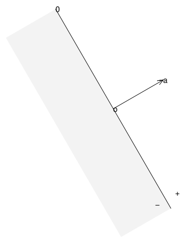
a is *normal vector*; x_0 lies on hyperplane

halfspace: $\{x \mid a^T x \leq b\}$ $(a \neq 0)$

convex; convex cone if $b = 0$

useful representation: $\{x \mid a^T(x - x_0) \leq 0\}$

a is *(outward) normal vector*; x_0 lies on boundary



Intersections

$$S_\alpha \text{ is } \begin{pmatrix} \textbf{subspace} \\ \textbf{affine} \\ \textbf{convex} \\ \textbf{convex cone} \end{pmatrix} \text{ for } \alpha \in \mathcal{A} \Longrightarrow \bigcap_{\alpha \in \mathcal{A}} S_\alpha \text{ is } \begin{pmatrix} \textbf{subspace} \\ \textbf{affine} \\ \textbf{convex} \\ \textbf{convex cone} \end{pmatrix}$$

example: *polyhedron* is intersection of a finite number of halfspaces

$$\begin{aligned} \mathcal{P} &= \{x \mid a_i^T x \leq b_i, \ \ i = 1, \ldots, k\} \\ &= \{x \mid Ax \preceq b\} \end{aligned}$$

($\preceq$ means componentwise)

a bounded polyhedron is called a *polytope*

in fact, *every* closed convex set S is (usually infinite) intersection of halfspaces:

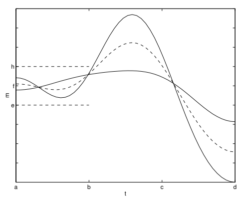
$$S = \cap \{\mathcal{H} \mid \mathcal{H} \text{ halfspace}, \ S \subseteq \mathcal{H}\}$$

(more later)

example:

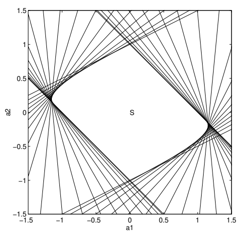
$$S = \{a \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

with $p(t) = \Sigma_{k=1}^m a_k \cos kt$



can express S as intersection of slabs: $S = \cap_{|t| \leq \pi/3} S_t$,

$$S_t = \left\{a \;\middle|\; -1 \leq \left[\, \cos t \ \cdots \ \cos mt \,\right] a \leq 1\right\}.$$



Combinations and hulls

$y = \theta_1 x_1 + \cdots + \theta_k x_k$ is a

- *linear combination* of $x_1, \ldots, x_k$
- *affine combination* if $\Sigma_i \theta_i = 1$
- *convex combination* if $\Sigma_i \theta_i = 1$, $\theta_i \geq 0$
- *conic combination* if $\theta_i \geq 0$

(linear,...) **hull** of S:
set of all (linear, ...) combinations from S

linear hull:	$\mathbf{Span}(S)$
affine hull:	$\mathbf{Aff}(S)$
convex hull:	$\mathbf{Co}\,(S)$
conic hull:	$\mathbf{Cone}(S)$

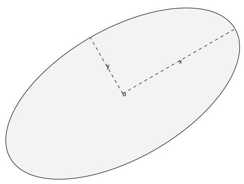
$$\mathbf{Co}\,(S) = \cap \{G \mid S \subseteq G, \ G \text{ convex } \}, \ldots$$

example. $S = \{(1,0,0), (0,1,0), (0,0,1)\}$
what is linear, affine, ..., hull?

Ellipsoids

$$\mathcal{E} = \{x \mid (x - x_c)^T A^{-1} (x - x_c) \leq 1\}$$

($A = A^T \succ 0$; $x_c \in \mathbf{R}^n$ center)



- semiaxis lengths: $\sqrt{\lambda_i}$; λ_i eigenvalues of A
- semiaxis directions: eigenvectors of A
- volume: $\alpha_n \left(\Pi\, \lambda_i\right)^{1/2} = \alpha_n \left(\det A\right)^{1/2}$

other descriptions

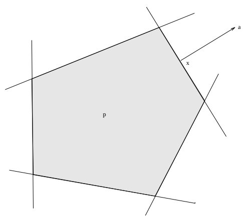
- $\mathcal{E} = \{Bu + x_c \mid \|u\| \leq 1\}$ ($\|u\| = \sqrt{u^T u}$)
- $\mathcal{E} = \{x \mid f(x) \leq 0\}$

$$\begin{aligned} f(x) &= x^T C x + 2 d^T x + e \\ &= \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} C & d \\ d^T & e \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \end{aligned}$$

($C = C^T \succ 0$, $e - d^T C^{-1} d < 0$)

exercise: convert among representations; give center, semiaxes, volume

Polyhedra



examples

- nonnegative orthant $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x \succeq 0\}$
- k-*simplex* **Co** $\{x_0, \ldots, x_k\}$ with x_0, …, x_k affinely independent, *i.e.*,

$$\textbf{Rank}\left(\begin{bmatrix} x_0 & x_1 & \cdots & x_k \\ 1 & 1 & \cdots & 1 \end{bmatrix}\right) = k + 1,$$

  or equivalently, $x_1 - x_0, \ldots, x_k - x_0$ lin. indep.

- *standard simplex* $\{x \in \mathbf{R}^n \mid x \succeq 0,\ \Sigma_i x_i = 1\}$, also called *probability simplex*

Norm balls & cones

$f : \mathbf{R}^n \to \mathbf{R}$ is a *norm* if for all $x, y \in \mathbf{R}^n$, $t \in \mathbf{R}$,

1. $f(x) \geq 0$; $f(x) = 0 \Longrightarrow x = 0$
2. $f(tx) = |t| f(x)$
3. $f(x + y) \leq f(x) + f(y)$

$f(x)$ usually denoted $\|x\|_{\mathrm{mark}}$ (subscript identifies norm)

if f is a norm,

- the *norm ball* $\mathsf{B} = \{x \mid f(x - x_c) \leq 1\}$ is convex
- the *norm cone* $C = \{(x, t) \mid f(x) \leq t\}$ is a convex cone

ℓ_p norms

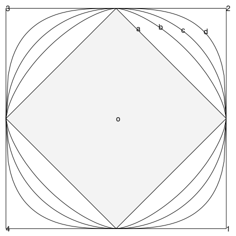
ℓ_p norms on $\mathbf{R}^n$: for $p \geq 1$,

$$\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p},$$

for $p = \infty$, $\|x\|_\infty = \max_i |x_i|$

- ℓ_2 norm is Euclidean norm $\|x\|_2 = \sqrt{\sum_i x_i^2}$
- ℓ_1 norm is sum-abs-values $\|x\|_1 = \sum_i |x_i|$
- ℓ_∞ norm is max-abs-value $\|x\|_\infty = \max_i |x_i|$

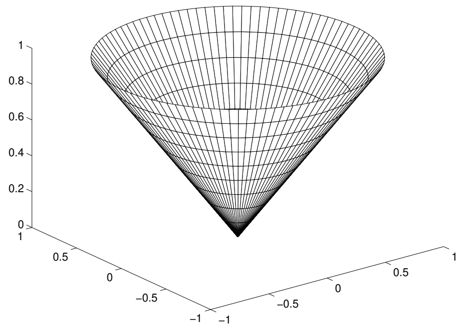
corresponding norm balls (in $\mathbf{R}^2$):



Second-order cone

norm cone associated with Euclidean norm is *second-order cone*
(also called *quadratic* or *Lorentz* cone)

$$\begin{aligned} S &= \{(x,t) \mid \sqrt{x^Tx} \leq t\} \\ &= \left\{(x,t) \;\middle|\; \begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq 0,\ t \geq 0 \right\} \end{aligned}$$



Affine transformations

suppose f is affine, *i.e.*, linear plus constant:

$$f(x) = Ax + b$$

if S, T convex, then so are

$$\begin{aligned} f^{-1}(S) &= \{x \mid Ax + b \in S\} \\ f(T) &= \{Ax + b \mid x \in T\} \end{aligned}$$

example: coordinate projection

$$\left\{x \;\middle|\; \begin{bmatrix} x \\ y \end{bmatrix} \in S \text{ for some } y \right\}$$

Linear matrix inequalities

$$\mathcal{P} = \{X \in \mathbf{R}^{n \times n} \mid X = X^T, X \succeq 0\}$$

is a convex cone, called the *positive semidefinite (PSD) cone* ($X \succeq 0$ means positive semidefinite)

$$\mathcal{P} = \bigcap_{z \in \mathbf{R}^n} \left\{X = X^T \;\middle|\; z^TXz = \sum_{i,j} z_i z_j X_{ij} \geq 0 \right\}$$

(intersection of infinite number of halfspaces in $\mathbf{R}^{n \times n}$)

hence, if $A_0, A_1, \ldots, A_m$ symmetric, the solution set of the *linear matrix inequality*

$$A_0 + x_1A_1 + \cdots + x_mA_m \succeq 0$$

is convex

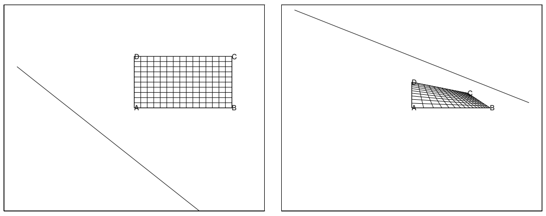
Linear-fractional transformation

linear-fractional (or projective) function $f : \mathbf{R}^m \to \mathbf{R}^n$,

$$f(x) = \frac{Ax+b}{c^T x + d}$$

on domain

$$\mathbf{dom}\, f = \mathcal{H} = \{x \mid c^T x + d > 0\}$$



line segments preserved: for x, $y \in \mathcal{H}$,

$$f([x,y]) = [f(x), f(y)]$$

hence, if C convex, $C \subseteq \mathcal{H}$, then $f(C)$ convex

Separating hyperplanes

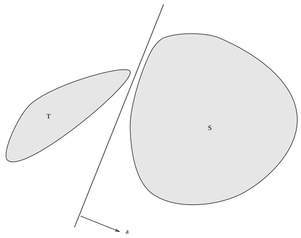
separating hyperplane theorem:

if S, $T \subseteq \mathbf{R}^n$ are convex and disjoint ($S \cap T = \emptyset$),

then, there are $a \neq 0$, b such that

$$a^T x \geq b \text{ for } x \in S, \quad a^T x \leq b \text{ for } x \in T$$

i.e., hyperplane $\{x \mid a^T x - b = 0\}$ *separates* S, T

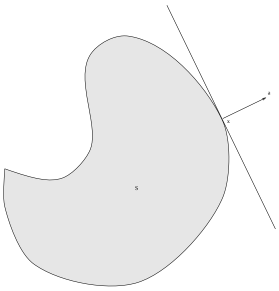


(stronger forms use strict inequality, require more conditions on S, T)

Supporting hyperplane

hyperplane $\{x \mid a^T x = a^T x_0\}$ *supports* S at $x_0 \in \partial S$ if

$$x \in S \Rightarrow a^T x \leq a^T x_0$$



halfspace $\{x \mid a^T x \leq b\}$ contains S for $b = a^T x_0$ but not for smaller b

S convex $\Rightarrow \exists$ supporting hyperplane for each $x_0 \in \partial S$

Generalized inequalities

suppose convex cone $K \subseteq \mathbf{R}^n$

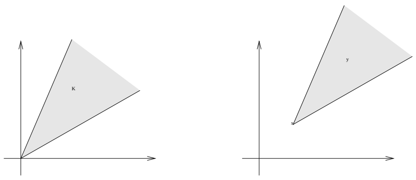
- is closed
- has nonempty interior
- is *pointed*: there is no line in K

K defines *generalized inequality* $\preceq_K$ in $\mathbf{R}^n$:

$$x \preceq_K y \iff y - x \in K$$

strict version:

$$x \preceq_K y \iff y - x \in \mathbf{Int} K$$



examples:

- $K = \mathbf{R}^n_+$: $x \preceq_K y$ means $x_i \leq y_i$ (componentwise vector inequality)
- K is PSD cone in $\{X \in \mathbf{R}^{n \times n} | X = X^T\}$: $X \preceq_K Y$ means $Y - X$ is PSD

(these are so common we drop K)

many properties of $\preceq_K$ similar to $\leq$ on $\mathbf{R}$, *e.g.*,

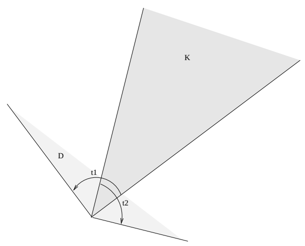
- $x \preceq_K y, \; u \preceq_K v \quad \Longrightarrow \quad x + u \preceq_K y + v$
- $x \preceq_K y, \; y \preceq_K x \quad \Longrightarrow \quad x = y$

unlike $\leq$, $\preceq_K$ is not in general a *linear ordering*

Dual cones and inequalities

if K is a cone, *dual cone* is defined as

$$K^\star = \{ \; y \mid x^T y \geq 0 \text{ for all } x \in K \; \}$$



- for $K = \mathbf{R}^n_+$, $K^\star = K$, since

$$\sum_i x_i y_i \geq 0 \text{ for all } x_i \geq 0 \quad \Longleftrightarrow \quad y_i \geq 0$$

- for $K =$ PSD cone, $K^\star = K$

(called *self-dual* cones)