Lecture 5
Extensions of convexity

– Quasiconvexity

– Log-convexity and variations

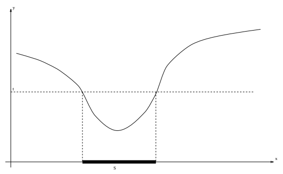
– K-convexity

Quasiconvex functions

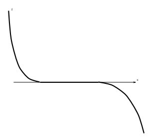
$f : \mathbf{R}^n \to \mathbf{R}$ is *quasiconvex* if every sublevel set

$$S_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \leq \alpha\}$$

is convex



can have 'locally flat' regions



f is *quasiconcave* if $-f$ is quasiconvex, *i.e.*, superlevel sets $\{x \mid f(x) \geq \alpha\}$ are convex

a function which is both quasiconvex and quasiconcave is called *quasilinear*

f convex (concave) $\Rightarrow$ f quasiconvex (quasiconcave)

Examples

– $f(x) = \sqrt{|x|}$ is quasiconvex on $\mathbf{R}$

– $f(x) = \log x$ is quasilinear on $\mathbf{R}_+$

– linear fractional function,

$$f(x) = \frac{a^T x + b}{c^T x + d}$$

is quasilinear on the halfspace $c^T x + d > 0$

– $f(x) = \frac{\|x-a\|}{\|x-b\|}$ is quasiconvex on the halfspace $\{x \mid \|x-a\| \leq \|x-b\|\}$

– maximum generalized eigenvalue of (X, Y), with $X = X^T$ and $Y = Y^T \succ 0$:

$$f(X,Y) = \min\{\lambda \mid \lambda Y - X \geq 0\}$$

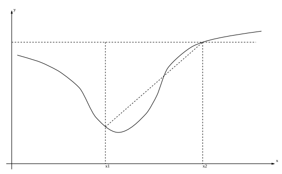
– $f(a) = \text{degree}(a_0 + a_1 t + \cdots + a_k t^k)$ on $\mathbf{R}^{k+1}$

Properties

- f is quasiconvex if and only if it is quasiconvex on lines, *i.e.*, $f(x_0 + th)$ quasiconvex in t for all x_0, h

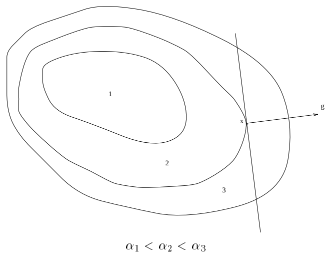
- modified Jensen's inequality: f is quasiconvex iff for all $x, y \in \textbf{dom}\, f$, $\theta \in [0, 1]$,

$$f(\theta x + (1-\theta)y) \le \max\{f(x), f(y)\}$$



- for f differentiable, f quasiconvex $\Longleftrightarrow$ for all $x, y \in \textbf{dom}\, f$

$$f(y) \le f(x) \Rightarrow (y-x)^T \nabla f(x) \le 0$$



- positive multiples

$$f \text{ quasiconvex}, \alpha \ge 0 \Longrightarrow \alpha f \text{ quasiconvex}$$

- pointwise maximum

$$f_1, f_2 \text{ quasiconvex } \Longrightarrow \max\{f_1, f_2\} \text{ quasiconvex}$$

(extends to supremum over arbitrary set)

- affine transformation of domain

$$f \text{ quasiconvex} \Longrightarrow f(Ax+b) \text{ quasiconvex}$$

- linear-fractional transformation of domain

$$f \text{ quasiconvex} \Longrightarrow f\left(\frac{Ax+b}{c^Tx+d}\right) \text{ quasiconvex}$$

on $c^Tx + d > 0$

- composition with monotone increasing function

$$f \text{ quasiconvex}, g \text{ monotone increasing}$$
$$\Longrightarrow g(f(x)) \text{ quasiconvex}$$

- sums of quasiconvex functions are **not** quasiconvex in general

- f quasiconvex in x, $y \Longrightarrow g(x) = \textbf{inf}_y\, f(x,y)$ quasiconvex in x

Nested sets characterization

f quasiconvex $\Rightarrow$ sublevel sets S_α are convex, nested, *i.e.*,

$$\alpha_1 \le \alpha_2 \Rightarrow S_{\alpha_1} \subseteq S_{\alpha_2}$$

converse: if T_α is a nested family of convex sets, then

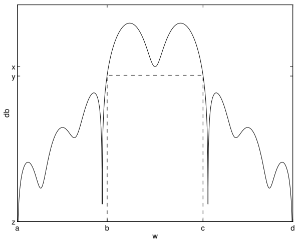
$$f(x) = \textbf{inf}\{\alpha \mid x \in T_\alpha\}$$

is quasiconvex

engineering interpretation: T_α are specs, tighter for smaller α

Examples

FIR filter: $H(\omega) = a_0 + \Sigma_{k=1}^{N} a_k \cos k\omega$



3dB-bandwidth

$$f(a) = \textbf{inf}\,\{\omega > 0 \mid 20\log_{10}(|H(\omega)|/|H(0)|) \leq -3.0\}$$

is a quasiconcave function on $\{a \in \mathbf{R}^{N+1} \mid H(0) > 0\}$

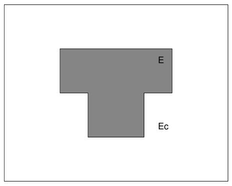
why? for $H(0) > 0$, $f(a) \geq \omega_0 \iff$

$$H(\omega) > H(0)/\sqrt{2} \ \text{ for } 0 \leq \omega < \omega_0$$

. . . an (infinite) intersection of halfspaces

electron-beam lithography

$E \subseteq [0,1] \times [0,1]$: desired exposure region
$E^c = [0,1] \times [0,1] \backslash E$: desired non-exposure region



$I(p)$: e-beam intensity at position $p \in [0,1] \times [0,1]$

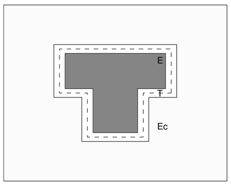
$$I(p) = \sum_i x_i g(p - p_i), \quad i = 1, \ldots, N$$

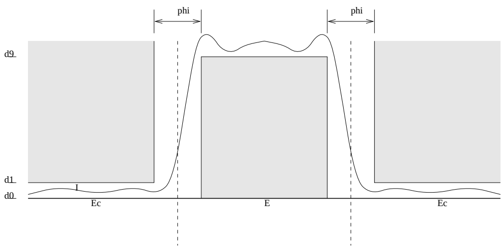
x_i: intensity of electron beam directed at pixel i
$g(p)$: given (point-spread) function

pattern transition width

define $\phi(x)$ as minimum α s.t.

$$\begin{array}{ll} I(p) \geq 0.9 & \text{for } \textbf{dist}\,(p, E^c) \geq \alpha \\ I(p) \leq 0.1 & \text{for } \textbf{dist}\,(p, E) \geq \alpha \end{array}$$





$\phi(x)$ is quasiconvex

Log-concave functions

$f : \mathbf{R}^n \to \mathbf{R}_+$ is log-concave (log-convex) if $\log f$ is concave (convex)

log-convex $\Rightarrow$ convex; concave $\Rightarrow$ log-concave

examples

- normal density, $f(x) = e^{-(1/2)(x-x_0)^T \Sigma^{-1}(x-x_0)}$
- erfc, $f(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$
- indicator function of convex set C:

$$I_C(x) = \begin{cases} 1 & x \in C \\ 0 & x \notin C \end{cases}$$

Properties

- sum of log-concave functions not always log-concave (but sum of log-convex functions **is** log-convex)
- products

  $$f, g \text{ log-concave } \implies fg \text{ log-concave}$$

  (immediate)
- integrals

  $f(x,y)$ log-concave in $x, y \Longrightarrow \int f(x,y)dy$ log-concave

  (not easy to show!)
- convolutions

  f, g log-concave $\Longrightarrow \int f(x-y)g(y)dy$ log-concave

  (immediate from the properties above)

Log-concave probability densities

many common probability density functions are log-concave

- normal ($\Sigma \succ 0$)

  $$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\bar{x})^T \Sigma^{-1}(x-\bar{x})}$$
- exponential ($\lambda_i > 0$)

  $$f(x) = \left(\prod_{i=1}^{n} \lambda_i\right) e^{-(\lambda_1 x_1 + \cdots + \lambda_n x_n)}$$

  on $\mathbf{R}_+^n$
- uniform distribution on convex (bounded) set C

  $$f(x) = \begin{cases} 1/\alpha & x \in C \\ 0 & x \notin C \end{cases}$$

  where α is Lebesgue measure of C (*i.e.*, length, area, volume ...)

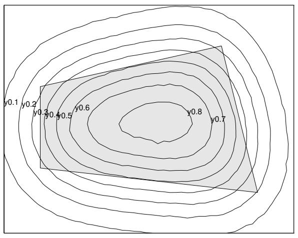
Example: manufacturing yield

$$x_{\mathsf{manu}} = x + v$$

- $x \in \mathbf{R}^n$: nominal value of design parameters
- $v \in \mathbf{R}^n$: manufacturing errors; zero mean random variable
- $S \subseteq \mathbf{R}^n$: specs, *i.e.*, acceptable values of x_{manu}

the *yield* $Y(x) = \mathbf{Prob}(x + v \in S)$ is log-concave if

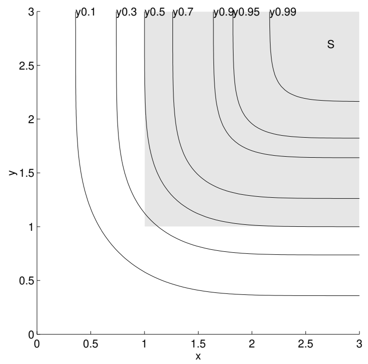
- S is a convex set
- the probability density of v is log-concave



example

- $S = \{x \in \mathbf{R}^2 \mid x_1 \geq 1, x_2 \geq 1\}$
- v_1, v_2: independent, normal with $\sigma = 1$

$$\begin{aligned} \text{yield}(x) &= \mathbf{Prob}(x + v \in S) \\ &= \frac{1}{2\pi}\left(\int_{1-x_1}^{\infty} e^{-t^2/2} dt\right)\left(\int_{1-x_2}^{\infty} e^{-t^2/2} dt\right) \end{aligned}$$



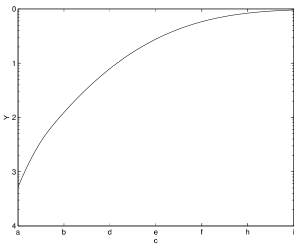
example (continued): max yield vs. cost

manufacturing cost $c = x_1 + 2x_2$

max yield for given cost is

$$Y^{\mathsf{opt}}(c) = \sup_{\substack{x_1+2x_2=c \\ x_1,x_2\geq 0}} Y(x)$$

Y^{opt} is log-concave



$$-\log Y^{\mathsf{opt}}(c) = \inf_{\substack{x_1+2x_2=c \\ x_1,x_2\geq 0}} -\log Y(x_1,x_2)$$

K-convexity

cvx. cone $K \subseteq \mathbf{R}^m$ induces generalized inequality $\preceq_K$

$f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is K-convex if $0 \leq \theta \leq 1 \implies$

$$f(\theta x + (1-\theta)y) \preceq_K \theta f(x) + (1-\theta)f(y)$$

example. K is PSD cone (called *matrix convexity*)

$f(X) = X^2$ is K-convex on $\{X \in \mathbf{R}^{m\times m} \mid X = X^T\}$,

let's show that for $\theta \in [0,1]$,

$$(\theta X + (1-\theta)Y)^2 \preceq \theta X^2 + (1-\theta)Y^2 \qquad (1)$$

for any $u \in \mathbf{R}^m$, $u^TX^2u = \|Xu\|^2$ is a (quadratic) convex fct of X, so

$$u^T(\theta X + (1-\theta)Y)^2u \leq \theta u^TX^2u + (1-\theta)u^TY^2u$$

which implies (1)