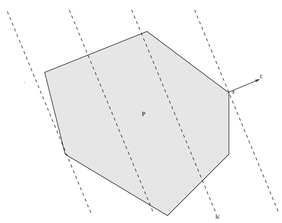
Lecture 8
Linear programming

- linear programming
- examples and applications
- linear fractional programming

Linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$



'standard' form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array}$$

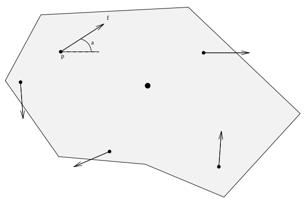
(widely used in LP literature & software)

variations, *e.g.*,

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

force/moment generation with thrusters

- rigid body with center of mass at origin $p = 0 \in \mathbf{R}^2$
- n forces with magnitude u_i, acting at $p_i = (p_{ix}, p_{iy})$, in direction θ_i



resulting horizontal force: $F_x = \Sigma_{i=1}^n u_i \cos\theta_i$

resulting vertical force: $F_y = \Sigma_{i=1}^n u_i \sin\theta_i$

resulting torque: $T = \Sigma_{i=1}^n (p_{iy} u_i \cos\theta_i - p_{ix} u_i \sin\theta_i)$

force limits: $0 \leq u_i \leq 1$ (thrusters)

fuel usage: $u_1 + \cdots + u_n$

problem: find thruster forces u_i that yield given desired forces and torques and minimize fuel usage
(if feasible)

can be expressed as LP:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & Fu = f^{\mathsf{des}} \\ & 0 \leq u_i \leq 1, \ i = 1, \ldots, n \end{array}$$

where

$$F = \begin{bmatrix} \cos\theta_1 & \cdots & \cos\theta_n \\ \sin\theta_1 & \cdots & \sin\theta_n \\ p_{1y}\cos\theta_1 - p_{1x}\sin\theta_1 & \cdots & p_{ny}\cos\theta_n - p_{nx}\sin\theta_n \end{bmatrix},$$

$$f^{\mathsf{des}} = (F_x^{\mathsf{des}},\ F_y^{\mathsf{des}},\ T^{\mathsf{des}}), \quad \mathbf{1} = (\ 1,\ 1,\ \cdots\ 1\ )$$

Converting LP to 'standard' form

– inequality constraints: write $a_i^T x \le b_i$ as

$$a_i^T x + s_i = b_i, \quad s_i \ge 0$$

s_i is called *slack variable* associated with $a_i^T x \le b_i$

– unconstrained variables: write $x_i \in \mathbf{R}$ as

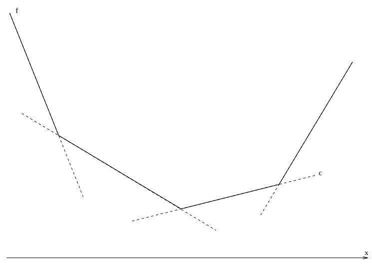
$$x_i = x_i^+ - x_i^-, \quad x_i^+, x_i^- \ge 0$$

Example. thruster problem in 'standard' form

$$\begin{array}{ll} \text{minimize} & [\mathbf{1}^T \; 0] \begin{bmatrix} u \\ s \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} u \\ s \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} F & 0 \\ I & I \end{bmatrix} \begin{bmatrix} u \\ s \end{bmatrix} = \begin{bmatrix} f^{\mathsf{des}} \\ \mathbf{1} \end{bmatrix} \end{array}$$

Piecewise-linear minimization

$$\begin{array}{ll} \text{minimize} & \max_i (c_i^T x + d_i) \\ \text{subject to} & Ax \preceq b \end{array}$$



express as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & c_i^T x + d_i \le t \\ & Ax \preceq b \end{array}$$

an LP in variables $x \in \mathbf{R}^n$, $t \in \mathbf{R}$

ℓ_∞- and ℓ_1-norm approximation

constrained ℓ_∞- (Chebychev) approximation

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_\infty \\ \text{subject to} & Fx \preceq g \end{array}$$

write as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \preceq t\mathbf{1} \\ & Ax - b \succeq -t\mathbf{1} \\ & Fx \preceq g \end{array}$$

constrained ℓ_1-approximation

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_1 \\ \text{subject to} & Fx \preceq g \end{array}$$

write as

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & Ax - b \preceq y \\ & Ax - b \succeq -y \\ & Fx \preceq g \end{array}$$

Extensions of thruster problem

– opposing thruster pairs

$$\begin{array}{ll} \text{minimize} & \Sigma_i |u_i| \\ \text{subject to} & Fu = f^{\mathsf{des}} \\ & |u_i| \le 1, \quad i = 1, \ldots, n \end{array}$$

can express as LP

– given f^{des},

$$\begin{array}{ll} \text{minimize} & \|Fu - f^{\mathsf{des}}\|_\infty \\ \text{subject to} & 0 \le u_i \le 1, \quad i = 1, \ldots, n \end{array}$$

can express as LP

– given f^{des},

$$\begin{array}{ll} \text{minimize} & \#\text{ thrusters on} \\ \text{subject to} & Fu = f^{\mathsf{des}} \\ & 0 \le u_i \le 1, \quad i = 1, \ldots, n \end{array}$$

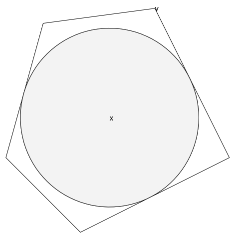
can't express as LP
(# thrusters on is quasiconcave!)

Design centering

find largest ball inside a polyhedron

$$\mathcal{P} = \{x \mid a_i^T x \le b_i,\ i = 1, \dots, m\}$$

center is called *Chebychev center*



ball $\{x_c + u \mid \|u\| \le r\}$ lies in $\mathcal{P}$ if and only if

$$\textbf{sup}\{a_i^T x_c + a_i^T u \mid \|u\| \le r\} \le b_i, \quad i = 1, \dots, m,$$

i.e.,

$$a_i^T x_c + r\|a_i\| \le b_i,\ i = 1, \dots, m$$

hence, finding Chebychev center is an LP:

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & a_i^T x_c + r\|a_i\| \le b_i, \quad i = 1, \dots, m \end{array}$$

Linear fractional program

$$\begin{array}{ll} \text{minimize} & \dfrac{c^T x + d}{f^T x + g} \\ \text{subject to} & Ax \preceq b \\ & f^T x + g > 0 \end{array}$$

- objective function is quasiconvex
- sublevel sets are polyhedra
- like LP, can be solved very efficiently

extension:

$$\begin{array}{ll} \text{minimize} & \max\limits_{i=1,\dots,K} \dfrac{c_i^T x + d_i}{f_i^T x + g_i} \\ \text{subject to} & Ax \preceq b \\ & f_i^T x + g_i > 0,\ i = 1, \dots, K \end{array}$$

- objective function is quasiconvex
- sublevel sets are polyhedra

Minimum-time optimal control

$$\begin{array}{l} x(t+1) = Ax(t) + Bu(t),\ t = 0, 1, \dots, K \\ x(0) = x_0 \\ \|u(t)\|_\infty \le 1,\ t = 0, 1, \dots, K \end{array}$$

variables: $u(0), \dots, u(K)$

settling time $f(u(0), \dots, u(K))$ is

$$\textbf{inf}\,\{T \mid x(t) = 0 \text{ for } T \le t \le K+1\}$$

f is quasiconvex function of $(u(0), \dots, u(K))$:

$$f(u(0), u(1), \dots, u(K)) \le T$$

if and only if for all $t = T, \dots, K+1$

$$x(t) = A^t x_0 + A^{t-1} B u(0) + \cdots + Bu(t-1) = 0$$

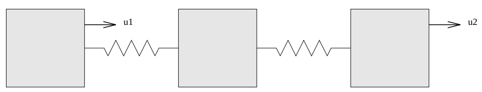
min-time optimal control problem:

$$\begin{array}{ll} \text{minimize} & f(u(0), u(1), \dots, u(K)) \\ \text{subject to} & \|u(t)\|_\infty \le 1,\ t = 0, \dots, K \end{array}$$

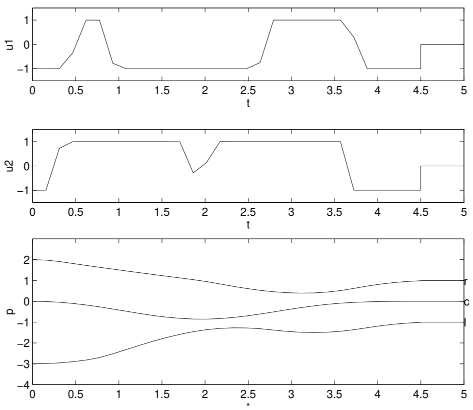
Min-time control example

three unit masses, connected by two unit springs with equilibrium length one

$u(t) \in \mathbf{R}^2$ is force on left & right mass over time interval $(0.15t, 0.15(t+1)]$

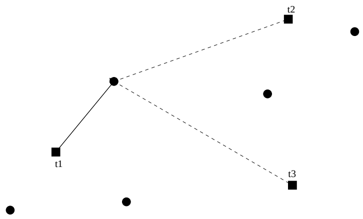


problem: pick $u(0), \dots, u(K)$ to bring masses to positions $(-1, 0, 1)$ (at rest), as quickly as possible, from initial condition $(-3, 0, 2)$ (at rest) optimal solution:



Optimal transmitter power allocation

- m transmitters, mn receivers all at same frequency
- transmitter i wants to transmit to n receivers labeled (i, j), $j = 1, \ldots, n$



- A_{ijk} is path gain from transmitter k to receiver (i, j)
- N_{ij} is (self) noise power of receiver (i, j)
- variables: transmitter powers p_k, $k = 1, \ldots, m$

at receiver (i, j):

- signal power:
$$S_{ij} = A_{iji} p_i$$
- noise plus interference power:
$$I_{ij} = \sum_{k \neq i} A_{ijk} p_k + N_{ij}$$
- signal to interference/noise ratio (SINR): S_{ij}/I_{ij}

problem: choose p_i to maximize smallest SINR:

$$\begin{array}{ll} \text{maximize} & \min_{i,j} \dfrac{A_{iji} p_i}{\sum_{k \neq i} A_{ijk} p_k + N_{ij}} \\ \text{subject to} & 0 \leq p_i \leq p_{\max} \end{array}$$

...a (generalized) linear fractional program

Nonconvex extensions of LP

Boolean LP or **zero-one LP**:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & Fx = g \\ & x_i \in \{0, 1\} \end{array}$$

integer LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & Fx = g \\ & x_i \in \mathbf{N} \end{array}$$

these are in general

- not convex problems
- **extremely difficult** to solve