Lecture 9
Convex quadratic programming

- quadratic optimization problems
- (quadratically constrained) quadratic programming
- second-order cone programming
- examples and applications

Quadratic functions and forms

- quadratic function

$$\begin{aligned} f(x) &= x^T P x + 2q^T x + r \\ &= \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P & q \\ q^T & r \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \end{aligned}$$

  convex if and only if $P \succeq 0$

- quadratic form $f(x) = x^T P x$
  convex if and only if $P \succeq 0$

- Euclidean norm $f(x) = \|Ax + b\|$
  (f^2 is a convex quadratic function ...)

Minimizing a quadratic function

$$\text{minimize } f(x) = x^T P x + 2q^T x + r$$

nonconvex case ($P \not\succeq 0$): unbounded below

proof: take $x = tv$, $t \to \infty$, where $Pv = \lambda v$, $\lambda < 0$

convex case ($P \succeq 0$): x is optimal if and only if

$$\nabla f(x) = 2Px + 2q = 0$$

two cases:

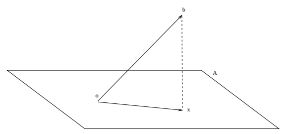
- $q \in \textbf{Range}(P)$: $f^\star > -\infty$
- $q \notin \textbf{Range}(P)$: unbounded below

important special case, $P \succ 0$:
unique optimal point $x_{\text{opt}} = -P^{-1}q$; $f^\star = r - q^T P^{-1} q$

Least-squares problems

minimize Euclidean norm
($A = [a_1 \cdots a_n]$ full rank, skinny)

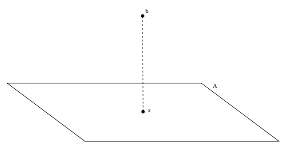
$$\text{minimize } \|Ax - b\|$$



solution: $x_{\text{ls}} = (A^T A)^{-1} A^T b$

minimum norm solution
(A full rank, fat)

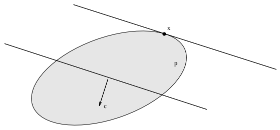
$$\begin{array}{ll} \text{minimize} & \|x\| \\ \text{subject to} & Ax = b \end{array}$$



solution: $x_{\text{mn}} = A^T (A A^T)^{-1} b$

Minimizing a linear function with quadratic constraint

minimize $c^T x$
subject to $x^T A x \leq 1$
($A = A^T \succ 0$)



$$x_{\mathrm{opt}} = -A^{-1}c/\sqrt{c^T A^{-1} c}$$

proof. Change of variables $y = A^{1/2}x$, $\tilde{c} = A^{-1/2}c$

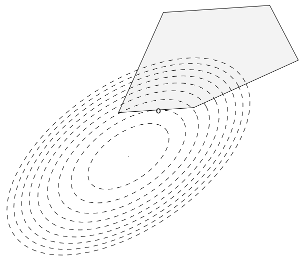
minimize $\tilde{c}^T y$
subject to $y^T y \leq 1$

optimal solution: $y_{\mathrm{opt}} = -\tilde{c}/\|\tilde{c}\|$

Quadratic program (QP)

quadratic objective, linear inequalities & equalities

minimize $x^T P x + 2q^T x + r$
subject to $Ax \preceq b, \quad Fx = g$



convex optimization problem if $P \succeq 0$

very hard problem if $P \not\succeq 0$

QCQP and SOCP

quadratically constrained quadratic programming (QCQP):

minimize $x^T P_0 x + 2q_0^T x + r_0$
subject to $x^T P_i x + 2q_i^T x + r_i \leq 0, \quad i = 1, \ldots, L$

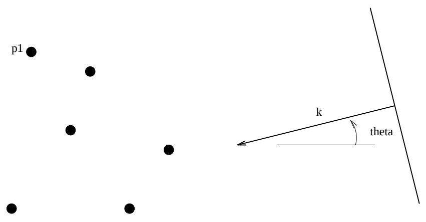
- convex if $P_i \succeq 0$, $i = 0, \ldots, L$
- nonconvex QCQP **very difficult**

second-order cone programming (SOCP):

minimize $c^T x$
subject to $\|A_i x + b_i\| \leq e_i^T x + d_i, \quad i = 1, \ldots, L$

includes QCQP (QP, LP)

Phased-array antenna beamforming



- omnidirectional antenna elements at positions $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$
- unit plane wave incident from angle θ induces in ith element a signal $e^{j(x_i \cos\theta + y_i \sin\theta - \omega t)}$
  ($j = \sqrt{-1}$, frequency ω, wavelength 2π)
- demodulate to get output $e^{j(x_i \cos\theta + y_i \sin\theta)} \in \mathbf{C}$
- linearly combine with complex weights w_i:

$$y(\theta) = \sum_{i=1}^{n} w_i e^{j(x_i \cos\theta + y_i \sin\theta)}$$

- $y(\theta)$ is (complex) *antenna array gain pattern*
- $|y(\theta)|$ gives sensitivity of array as function of incident angle θ
- depends on design variables $\mathbf{Re}\, w$, $\mathbf{Im}\, w$
  (called *antenna array weights* or *shading coefficients*)

design problem: choose w to achieve desired gain pattern

Sidelobe level minimization

make $|y(\theta)|$ small for $|\theta - \theta_{\mathsf{tar}}| > \alpha$

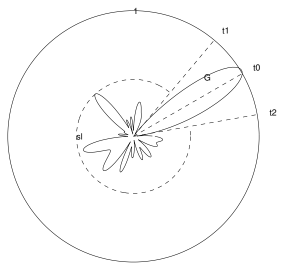
(θ_{tar}: target direction; 2α: beamwidth)

via least-squares (discretize angles)

$$\begin{array}{ll} \text{minimize} & \Sigma_i |y(\theta_i)|^2 \\ \text{subject to} & y(\theta_{\mathsf{tar}}) = 1 \end{array}$$

(sum over angles outside beam)

least-squares problem with two (real) linear equality constraints



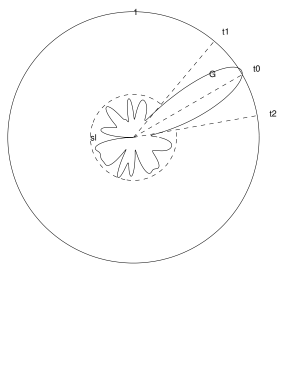
minimize sidelobe level (discretize angles)

$$\begin{array}{ll} \text{minimize} & \max_i |y(\theta_i)| \\ \text{subject to} & y(\theta_{\mathsf{tar}}) = 1 \end{array}$$

(max over angles outside beam)

can be cast as SOCP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & |y(\theta_i)| \le t \\ & y(\theta_{\mathsf{tar}}) = 1 \end{array}$$



Extensions

convex (& quasiconvex) extensions:

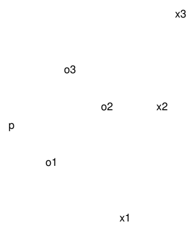
- $y(\theta_0) = 0$ (null in direction θ_0)
- w is real (amplitude only shading)
- $|w_i| \le 1$ (attenuation only shading)
- minimize $\sigma^2 \Sigma_{i=1}^n |w_i|^2$ (thermal noise power in y)
- minimize beamwidth given a maximum sidelobe level

nonconvex extension:

- maximize number of zero weights

Optimal receiver location

N transmitter frequencies $1, \ldots, N$
transmitters @ a_i, b_i use frequency i (a_i, $b_i \in \mathbf{R}^2$)
transmitters @ a_1, a_2, $\ldots$, a_N are wanted
transmitters @ b_1, b_2, $\ldots$, b_N are interfering



(signal) receiver power from a_i: $\|x - a_i\|^{-\alpha}$

(interfering) receiver power from b_i: $\|x - b_i\|^{-\alpha}$
$(\alpha \approx 2.1)$

worst signal to interference ratio as a function of receiver position x:

$$\mathsf{S/I} = \min_i \frac{\|x - a_i\|^{-\alpha}}{\|x - b_i\|^{-\alpha}}$$

what is the optimal receiver location?