Lecture 6
FIR filter design

- FIR filters
- linear phase filter design
- magnitude filter design
- equalizer design

FIR filters

finite impulse response (FIR) filter:

$$y(t) = \sum_{\tau=0}^{n-1} h_\tau u(t-\tau), \quad t \in \mathbf{Z}$$

- (sequence) $u : \mathbf{Z} \to \mathbf{R}$ is *input signal*
- (sequence) $y : \mathbf{Z} \to \mathbf{R}$ is *output signal*
- $h_i \in \mathbf{R}$ are called *filter coefficients*
- n is filter *order* or *length*

filter frequency response: $H : \mathbf{R} \to \mathbf{C}$

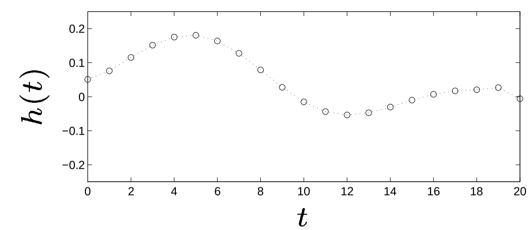
$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-j\omega} + \cdots + h_{n-1} e^{-j(n-1)\omega} \\ &= \sum_{t=0}^{n-1} h_t \cos t\omega - j \sum_{t=0}^{n-1} h_t \sin t\omega \end{aligned}$$

- j means $\sqrt{-1}$ here (EE tradition)
- H is periodic and conjugate symmetric, so only need to know/specify for $0 \le \omega \le \pi$

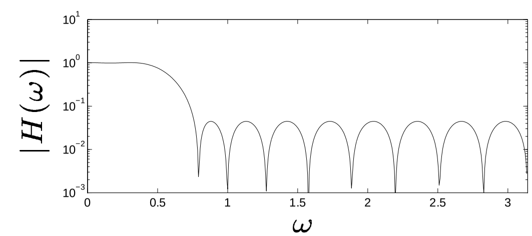
FIR filter design problem:
choose h so H and h satisfy/optimize specs

example: (lowpass) FIR filter, order $n = 21$

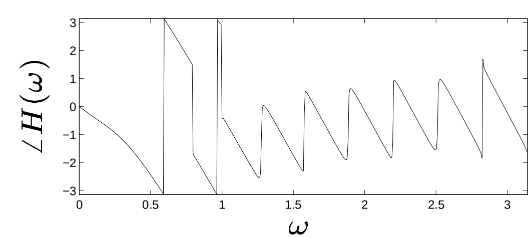
impulse response h:



frequency response magnitude ($i.e.$, $|H(\omega)|$):



frequency response phase ($i.e.$, $\angle H(\omega)$):



Linear phase filters

suppose

- $n = 2N + 1$ is odd
- impulse response is symmetric about midpoint:

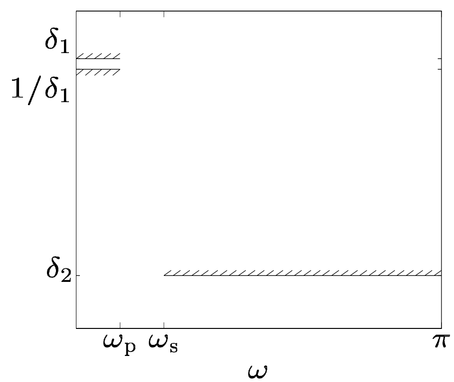
$$h_t = h_{n-1-t}, \quad t = 0, \ldots, n-1$$

then

$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-j\omega} + \cdots + h_{n-1} e^{-j(n-1)\omega} \\ &= e^{-jN\omega} \left(2h_0 \cos N\omega + 2h_1 \cos(N-1)\omega + \cdots + h_N \right) \\ &\stackrel{\Delta}{=} e^{-jN\omega} \widetilde{H}(\omega) \end{aligned}$$

- term $e^{-jN\omega}$ represents N-sample delay
- $\widetilde{H}(\omega)$ is **real**
- $|H(\omega)| = |\widetilde{H}(\omega)|$
- called **linear phase** filter ($\angle H(\omega)$ is linear except for jumps of $\pm\pi$)

Lowpass filter specifications



idea:

- pass frequencies in *passband* $[0, \omega_{\rm p}]$
- block frequencies in *stopband* $[\omega_{\rm s}, \pi]$

specifications:

- max. *passband ripple* ($\pm 20 \log_{10} \delta_1$ in dB):

$$1/\delta_1 \leq |H(\omega)| \leq \delta_1, \quad 0 \leq \omega \leq \omega_{\rm p}$$

- min. *stopband attenuation* ($-20 \log_{10} \delta_2$ in dB):

$$|H(\omega)| \leq \delta_2, \quad \omega_{\rm s} \leq \omega \leq \pi$$

Linear phase lowpass filter design

- sample frequency ($\omega_k = k\pi/K$, $k = 1, \ldots, K$)
- can assume wlog $\widetilde{H}(0) > 0$, so ripple spec is

$$1/\delta_1 \leq \widetilde{H}(\omega_k) \leq \delta_1$$

design for **maximum stopband attenuation:**

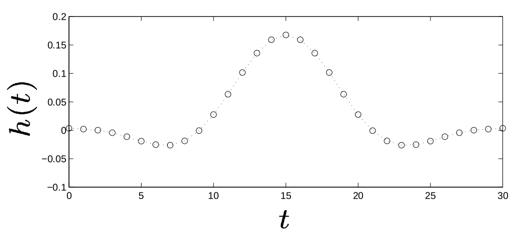
$$\begin{array}{ll} \mbox{minimize} & \delta_2 \\ \mbox{subject to} & 1/\delta_1 \leq \widetilde{H}(\omega_k) \leq \delta_1, \quad 0 \leq \omega_k \leq \omega_{\rm p} \\ & -\delta_2 \leq \widetilde{H}(\omega_k) \leq \delta_2, \quad \omega_{\rm s} \leq \omega_k \leq \pi \end{array}$$

- passband ripple δ_1 is given
- an LP in variables h, δ_2
- known (and used) since 1960's
- can add other constraints, *e.g.*, $|h_i| \leq \alpha$

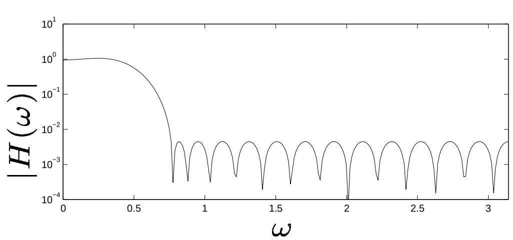
example

- linear phase filter, $n = 31$
- passband $[0, 0.12\pi]$; stopband $[0.24\pi, \pi]$
- max ripple $\delta_1 = 1.059$ (± 0.5dB)
- design for maximum stopband attenuation

impulse response h:



frequency response magnitude ($i.e.$, $|H(\omega)|$):



Some variations

$$\widetilde{H}(\omega) = 2h_0 \cos N\omega + 2h_1 \cos(N-1)\omega + \cdots + h_N$$

minimize passband ripple (given δ_2, ω_s, ω_p, N)

$$\begin{array}{ll} \text{minimize} & \delta_1 \\ \text{subject to} & 1/\delta_1 \le \widetilde{H}(\omega_k) \le \delta_1, \quad 0 \le \omega_k \le \omega_{\rm p} \\ & -\delta_2 \le \widetilde{H}(\omega_k) \le \delta_2, \quad \omega_{\rm s} \le \omega_k \le \pi \end{array}$$

minimize transition bandwidth (given $\delta_1, \delta_2, \omega_s, N$)

$$\begin{array}{ll} \text{minimize} & \omega_{\rm s} \\ \text{subject to} & 1/\delta_1 \le \widetilde{H}(\omega_k) \le \delta_1, \quad 0 \le \omega_k \le \omega_{\rm p} \\ & -\delta_2 \le \widetilde{H}(\omega_k) \le \delta_2, \quad \omega_{\rm s} \le \omega_k \le \pi \end{array}$$

minimize filter order (given δ_1, δ_2, ω_s, ω_p)

$$\begin{array}{ll} \text{minimize} & N \\ \text{subject to} & 1/\delta_1 \le \widetilde{H}(\omega_k) \le \delta_1, \quad 0 \le \omega_k \le \omega_{\rm p} \\ & -\delta_2 \le \widetilde{H}(\omega_k) \le \delta_2, \quad \omega_{\rm s} \le \omega_k \le \pi \end{array}$$

- can be solved using bisection
- each iteration is an LP feasibility problem

Filter magnitude specifications

transfer function *magnitude spec* has form

$$L(\omega) \le |H(\omega)| \le U(\omega), \quad \omega \in [0, \pi]$$

where $L, U : \mathbf{R} \to \mathbf{R}_+$ are given and

$$H(\omega) = \sum_{t=0}^{n-1} h_t \cos t\omega - j \sum_{t=0}^{n-1} h_t \sin t\omega$$

- arises in many applications, $e.g.$, audio, spectrum shaping
- not equivalent to a set of linear inequalities in h (lower bound is not even convex)
- can change variables and convert to set of linear inequalities

Autocorrelation coefficients

autocorrelation coefficients associated with impulse response $h = (h_0, \ldots, h_{n-1}) \in \mathbf{R}^n$ are

$$r_t = \sum_\tau h_\tau h_{\tau+t}$$

(we take $h_k = 0$ for $k < 0$ or $k \ge n$)

- $r_t = r_{-t}$; $r_t = 0$ for $|t| \ge n$
- hence suffices to specify $r = (r_0, \ldots, r_{n-1}) \in \mathbf{R}^n$

Fourier transform of autocorrelation coefficients is

$$R(\omega) = \sum_\tau e^{-j\omega\tau} r_\tau = r_0 + \sum_{t=1}^{n-1} 2 r_t \cos \omega t = |H(\omega)|^2$$

- always have $R(\omega) \ge 0$ for all ω
- can express magnitude specification as

$$L(\omega)^2 \le R(\omega) \le U(\omega)^2, \quad \omega \in [0, \pi]$$

. . . linear inequalities in r

Spectral factorization

question: when is $r \in \mathbf{R}^n$ the autocorrelation coefficients of some $h \in \mathbf{R}^n$?

answer (*spectral factorization theorem*):
if and only if $R(\omega) \geq 0$ for all ω

- spectral factorization condition is convex in r (a linear inequality for each ω)
- many algorithms for spectral factorization, $i.e.$, finding an h s.t. $R(\omega) = |H(\omega)|^2$

magnitude design via autocorrelation coefficients:

- use r as variable (instead of h)
- add spec. fact. condition $R(\omega) \geq 0$ for all ω
- optimize over r
- use spectral factorization to recover h

Magnitude lowpass filter design

maximum stopband attenuation design:

with variables r and $\tilde{\delta}_2$, problem becomes

$$\begin{array}{ll} \text{minimize} & \tilde{\delta}_2 \\ \text{subject to} & 1/\tilde{\delta}_1 \leq R(\omega) \leq \tilde{\delta}_1, \quad \omega \in [0, \omega_{\rm p}] \\ & R(\omega) \leq \tilde{\delta}_2, \quad \omega \in [\omega_{\rm s}, \pi] \\ & R(\omega) \geq 0, \quad \omega \in [0, \pi] \end{array}$$

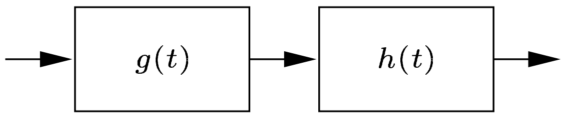
($\tilde{\delta}_i$ corresponds to δ_i^2 in original problem)

discretize frequency:

$$\begin{array}{ll} \text{minimize} & \tilde{\delta}_2 \\ \text{subject to} & 1/\tilde{\delta}_1 \leq R(\omega_k) \leq \tilde{\delta}_1, \quad 0 \leq \omega_k \leq \omega_{\rm p} \\ & R(\omega_k) \leq \tilde{\delta}_2, \quad \omega_{\rm s} \leq \omega_k \leq \pi \\ & R(\omega_k) \geq 0, \quad 0 \leq \omega_k \leq \pi \end{array}$$

. . . an LP in r, $\tilde{\delta}_2$

Equalizer design



(time-domain) equalization: given

- g (unequalized impulse response)
- g_{des} (desired impulse response)

design (FIR equalizer) h so that

$$\tilde{g} \triangleq h * g \approx g_{\mathrm{des}}$$

common choice: pure delay D

$$g_{\mathrm{des}}(t) = \begin{cases} 1 & t = D \\ 0 & t \neq D \end{cases}$$

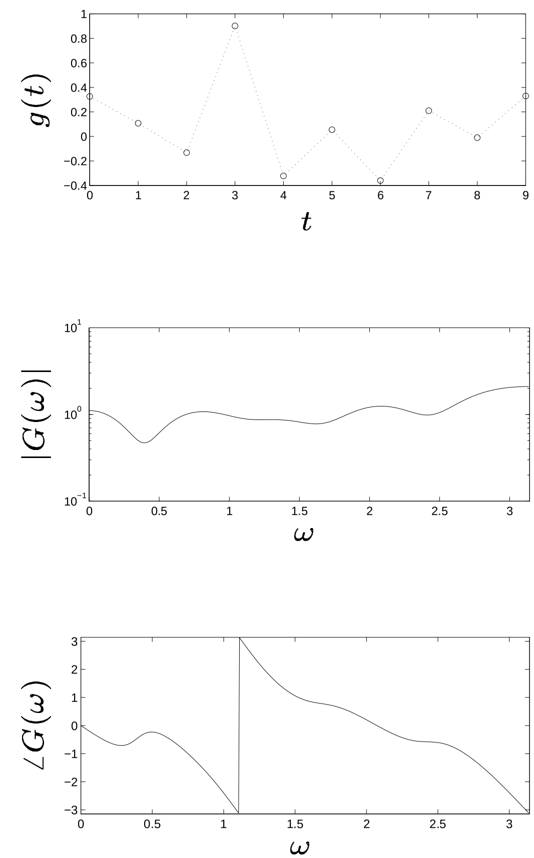
as an LP:

$$\begin{array}{ll} \text{minimize} & \max_{t \neq D} |\tilde{g}(t)| \\ \text{subject to} & \tilde{g}(D) = 1 \end{array}$$

can use $\sum_{t \neq D} |\tilde{g}(t)|$ or $\sum_{t \neq D} \tilde{g}(t)^2$

example

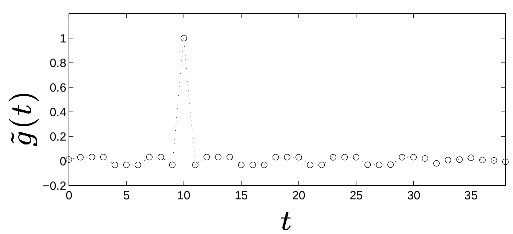
unequalized system G is 10th order FIR:



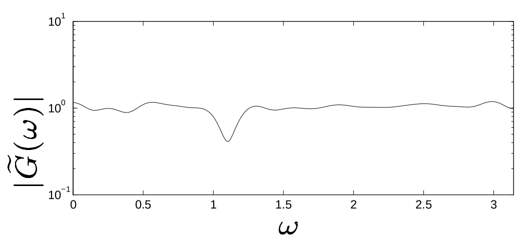
design 30th order FIR equalizer with $\widetilde{G}(\omega) \approx e^{-j10\omega}$

$$\text{minimize} \ \max_{t \neq 10} |\tilde{g}(t)|$$

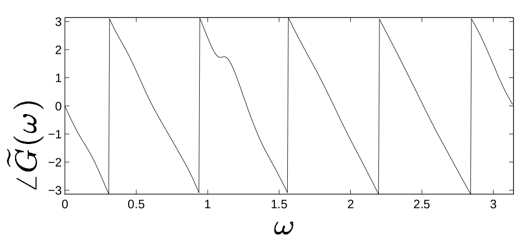
equalized system impulse response $\tilde{g}$



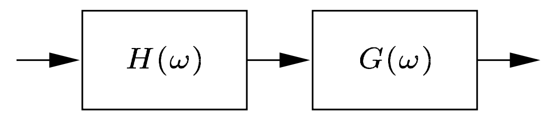
equalized frequency response magnitude $|\widetilde{G}|$



equalized frequency response phase $\angle\widetilde{G}$



Magnitude equalizer design



- given system frequency response $G : [0,\pi] \to \mathbf{C}$
- design FIR equalizer H so that $|G(\omega)H(\omega)| \approx 1$:

$$\text{minimize} \ \max_{\omega\in[0,\pi]} \left| |G(\omega)H(\omega)|^2 - 1 \right|$$

use autocorrelation coefficients as variables:

$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & \left| |G(\omega)|^2 R(\omega) - 1 \right| \le \alpha, \ \ \omega \in [0,\pi] \\ & R(\omega) \ge 0, \ \ \omega \in [0,\pi] \end{array}$$

when discretized, an LP in r, α, . . .

Multi-system magnitude equalization

- given M frequency responses $G_k : [0,\pi] \to \mathbf{C}$
- design FIR equalizer H so that $|G_k(\omega)H(\omega)| \approx$ constant:

$$\begin{array}{ll} \text{minimize} & \displaystyle\max_{k=1,\ldots,M} \max_{\omega\in[0,\pi]} \left| |G_k(\omega)H(\omega)|^2 - \gamma_k \right| \\ \text{subject to} & \gamma_k \geq 1, \quad k = 1,\ldots,M \end{array}$$

use autocorrelation coefficients as variables:

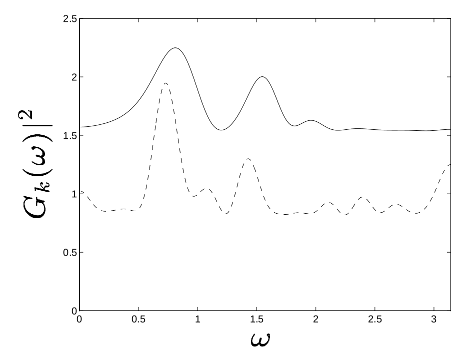
$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & \left| |G_k(\omega)|^2 R(\omega) - \gamma_k \right| \leq \alpha, \\ & \qquad \omega \in [0,\pi], \quad k = 1,\ldots,M \\ & R(\omega) \geq 0, \quad \omega \in [0,\pi] \\ & \gamma_k \geq 1, \quad k = 1,\ldots,M \end{array}$$

. . . when discretized, an LP in γ_k, r, α

example

- $M = 2$, $n = 25$
- $\gamma_k \geq 1$

unequalized frequency responses



equalized frequency responses