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THE SHAPE OF THE TALLEST COLUMN*

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Abstract. The height at which an unloaded column will buckle under its own weight is the fourth root of the least eigenvalue of a certain Sturm–Liouville operator. We show that the operator associated with the column proposed by Keller and Niordson [*J. Math. Mech.*, 16 (1966), pp. 433–446] *does not* possess a discrete spectrum. This calls into question their formal use of perturbation theory, so we consider a class of designs that permits a tapered summit yet still guarantees a discrete spectrum. Within this class we prove that the least eigenvalue increases when one replaces a design with its decreasing rearrangement. This leads to a very simple proof of the existence of a tallest column.

Key words. buckling load, self-weight, continuous spectrum, rearrangement

AMS subject classifications. 34L15, 49J99, 73H05

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1. Introduction.

Euler [4], [5] posed and solved the problem of buckling of prismatic columns under self-weight. He found that a column, clamped at its base and free at its summit, could be built to a height of

$$H_c = \left(\frac{9EI}{4\rho A} j_{-1/3}^2 \right)^{1/3}$$

before buckling under its own weight. Here E denotes Young's modulus, ρ denotes weight density, A and I denote cross-sectional area and its second moment, and $j_{-1/3} \approx 1.8663$ is the least positive root of the Bessel function of order $-1/3$. To take a particular instance, the critical height of a circular cylinder of volume V is

$$H_c = \left(\frac{9EV}{16\pi\rho} j_{-1/3}^2 \right)^{1/4}.$$

Almost two hundred years elapsed before Keller and Niordson [9] asked what height one could reach if, while fixing V, the overall volume, one was permitted to taper the column by varying A, and hence I, from point to point. Keller and Niordson formulated the Euler problem of critical height in terms of an eigenvalue problem for an ordinary differential operator and proceeded to maximize its least eigenvalue over a large class of shapes. In the spirit of Keller's previous attacks on buckling problems, [8] and [14], it was supposed that this least eigenvalue varied smoothly over the admissible class of shapes. This approach led Keller and Niordson to propose a shape that is so severely tapered at its summit that we are able to show that the associated differential operator *does not* possess a discrete spectrum. As this calls into doubt their method, if not their result, we argue that the problem merits reconsideration. Indeed, though the literature on column buckling is vast, see, e.g., Gajewski and Zyczkowski [7], the tallest column problem appears to have started and ended with the work of Keller and Niordson!

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In section 2 we recall Keller and Niordson's formulation of the problem, their proposed tallest column, and present a proof that the associated spectrum is not discrete. In section 3 we isolate an admissible class of designs that permits tapered ends and guarantees a discrete spectrum. Within this class we prove, in section 4, the intuitively obvious, though technically elusive, fact that the least eigenvalue increases when one replaces a shape with its decreasing rearrangement. This leads to a very simple proof, in section 5, of the existence of a tallest column.

2. The work of Keller and Niordson. Tapered cross sections appear through the dependence of A on z, the distance from the clamped base. Assuming simply connected, geometrically similar cross sections, we find $I(z) = \alpha A^2(z)$, where α is a geometric constant. If $y(z)$ is the lateral deflection, from vertical, of the cross section at z and $EI(z)y''(z)$ is the associated bending moment then a balance of moments brings

$$E\alpha A^2(z)y''(z) = \int_z^H \rho A(\tilde{z})[y(\tilde{z}) - y(z)]\, d\tilde{z}, \quad 0 < z < H, \tag{2.1}$$

where H is the height of the column. Clamping the column at its base is synonymous with $y(0) = y'(0) = 0$. With V denoting the column's volume, Keller and Niordson introduce the dimensionless variables

$$x = z/H, \quad a(x) = HA(xH)/V, \quad \eta(x) = y(xH)/H, \quad \lambda = \rho H^4/\alpha EV$$

and so arrive at

$$a^2(x)\eta'' = \lambda \int_x^1 a(\tilde{x})[\eta(\tilde{x}) - \eta(x)]\, d\tilde{x}, \quad \eta(0) = \eta'(0) = 0, \tag{2.2}$$

and the area normalization

$$\int_0^1 a\, dx = 1. \tag{2.3}$$

Differentiating (2.2) with respect to x and calling $u(x) = \eta'(x)$, they finally obtain

$$-(a^2(x)u'(x))' = \lambda \left(\int_x^1 a(t)\, dt \right) u(x), \quad 0 < x < 1, \quad u(0) = a^2(1)u'(1) = 0. \tag{2.4}$$

We shall refer to this eigenvalue problem as the Euler problem and denote by $\lambda_1(a)$ its least eigenvalue. Keller and Niordson took up the problem of maximizing $\lambda_1(a)$ over those nonnegative a satisfying the volume constraint, (2.3). Supposing the existence of an optimal design, formal perturbation theory led them to a candidate $\tilde{a}$ for which

$$\tilde{a}(x) = \begin{cases} O(1), & \text{as } x \to 0, \\ c(1-x)^3 + O((1-x)^4), & \text{as } x \to 1, \end{cases} \tag{2.5}$$

where c is a positive constant. We shall now argue that the spectrum of the Euler problem is not discrete for such a design.

Following Friedrichs [6], we consider

$$\frac{1}{Z(x)} \equiv 4\tilde{a}^2(x) \left(\int_x^1 \tilde{a}(t)\, dt \right) \left(\int_0^x \tilde{a}(t)^{-2}\, dt \right)^2 \tag{2.6}$$

and recall [6, Criteria II & III] that if

$$\lambda_* \equiv \lim_{x\to 1} Z(x)$$

exists and

$$\lambda_* \le \liminf_{x\to 0} Z(x)$$

then the spectrum of (2.4) is discrete below λ_* and nondiscrete above λ_*. On substituting (2.5) into (2.6) we find, near $x = 1$, that

$$\begin{aligned}\frac{1}{Z(x)} &= 4\left(c^2(1-x)^6 + O((1-x)^7)\right)\left(\frac{c(1-x)^4}{4} + O((1-x)^5)\right) \\ &\qquad\qquad \times \left(\frac{1}{5c^2(1-x)^5} + \frac{1}{O((1-x)^4)}\right)^2 \\ &= \frac{1}{25c} + O(1-x)\end{aligned}$$

and so $\lambda_* = 25c$. As $\tilde{a}$ is well behaved near $x = 0$, it follows easily that

$$\liminf_{x\to 0} Z(x) = +\infty.$$

We have just established the following proposition.

PROPOSITION 2.1. *The spectrum of the Euler problem,* (2.4), *with the Keller–Niordson design, $\tilde{a}$, is discrete below* $25c$ *and nondiscrete above* $25c$.

We note that the result states that *if* (2.4) has spectrum below $25c$ then this spectrum is discrete. It remains to see whether any such design produces eigenvalues below $25c$. Keller and Niordson's calculation of $c = \lambda_1/24$ suggests that their design indeed gives rise to an isolated eigenvalue, $\lambda_1 = 24c$, just below the continuous spectrum. Their result however was predicated on the false assumption that (2.4) possessed a purely discrete spectra. Nevertheless, we now construct a concrete design of unit volume that satisfies (2.5) and possesses at least one eigenvalue below $25c$. Again, the main idea lies with Friedrichs [6]; if there exists a function u for which $u(0) = 0$ and

$$\int_0^1 \tilde{a}^2 |u'|^2\,dx < \lambda_* \int_0^1 \left(\int_x^1 \tilde{a}(t)\,dt\right) u^2\,dx < \infty$$

then (2.4) possesses at least one eigenvalue below λ_*. We shall apply this to

$$\tilde{a}(x) = 5(1-x)^3 - (5/4)(1-x)^4 \quad \text{and} \quad u(x) = \frac{x}{(1-x)^2}.$$

This $\tilde{a}$ is clearly positive away from $x = 1$ and satisfies the volume constraint (2.3). These choices produce $\lambda_* = 125$,

$$\int_0^1 \tilde{a}^2|u'|^2\,dx = \frac{1145}{24}, \quad \text{and} \quad \int_0^1 \left(\int_x^1 \tilde{a}(t)\,dt\right) u^2\,dx = \frac{19}{48},$$

and so this design has an eigenvalue below

$$\frac{1145/24}{19/48} \approx 120.5263.$$

Although one may easily generalize this example we have not been able to produce an exact characterization of those $\tilde{a}$ that satisfy (2.5) *and* possess at least one isolated eigenvalue. In other words, we have not been able to formulate a (large) class of admissible designs that accommodate Keller and Niordson's belief in cubic taper and presence of discrete spectra. In the interest of rigorously establishing the existence of a tallest column we have been compelled to exclude the possibility of cubic taper.

3. The Green's function. When the Green's function associated with the Euler problem is square integrable, the associated Green's operator is compact on $L^2(0,1)$ and therefore in possession of a discrete spectrum.

Following the standard recipe, see, e.g., Porter and Stirling [12, Example 6.13], the Green's function associated with the Euler problem (2.4) is

$$g(x,y;a) = \sqrt{w(x)w(y)} \int_0^{x\wedge y} \frac{dt}{a^2(t)},$$

where $x \wedge y = \min\{x,y\}$ and

$$w(x) \equiv \int_x^1 a(t)\,dt.$$

We now isolate a class of a that is rich enough to describe a large number of columns yet narrow enough to guarantee purely discrete spectra. On physical grounds it seems apparent that the tallest columns will be those that taper at their summit. We control the degree of taper by asking the cross-sectional area to lie in

$$ad_p \equiv \{a : k_1(1-x)^p \le a(x) \le k_2(1-x)^p\}$$

for some positive values of $k_1,\ k_2$, and p. It follows immediately that for such a

$$g^2(x,y;a) \le \frac{k_2^2}{k_1^2(1+p)^2(1-2p)^2}(1-x)^{1+p}(1-y)^{1+p}(1+(1-x\wedge y)^{2-4p}).$$

This being integrable over $(0,1)\times(0,1)$ so long as $0 \le p < 3$, we find the following proposition.

PROPOSITION 3.1. *If $a \in ad_p$ and $0 \le p < 3$ then the spectrum of the Euler problem is discrete.*

Proof. As $g(\cdot,\cdot;a) \in L^2((0,1)\times(0,1))$, it follows from [12, Theorem 3.4] that the Green's operator

$$G(a)\phi(x) \equiv \int_0^1 g(x,y;a)\phi(y)\,dy$$

is a compact operator on $L^2(0,1)$. This operator is also self-adjoint and positive and so, see, e.g., [12, Theorem 4.15], its spectrum is composed solely of a discrete sequence of nonnegative real numbers. ∎

We remark that $\tilde{a}$, the design of Keller and Niordson, remains just out of our reach. We next invoke [12, Lemma 5.1] in the variational characterization

$$\frac{1}{\lambda_1(a)} = \max_{\|\phi\|=1} \langle G(a)\phi, \phi\rangle, \tag{3.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard $L^2(0,1)$ inner product and $\|\cdot\|$ denotes the associated norm. The maximum is attained at $\phi_1 = \sqrt{w}u_1$, where u_1 is the first eigenfunction of

(2.4). As the boundary conditions on (2.4) are separated, the standard oscillation theory implies that $\lambda_1(a)$ is simple and that u_1 may be assumed everywhere nonnegative. Our first application of (3.1) is the following proposition.

PROPOSITION 3.2. *If $a \in ad_p$ and $0 \leq p < 3$ and a satisfies the volume constraint* (2.3) *then*

$$\lambda_1(a) \leq \frac{720}{k_1}.$$

Proof. Choosing $\phi \equiv 1$ in (3.1) brings

$$\frac{1}{\lambda_1(a)} \geq \int_0^1 \int_0^1 g(x, y; a)\, dx\, dy,$$

so we proceed to establish a pointwise lower bound for g.

$$\begin{aligned} g(x, y; a) &\geq \frac{k_1}{4}(1-x)^2(1-y)^2(x \wedge y)\left(\frac{1}{x \wedge y}\int_0^{x \wedge y} a^{-2}(t)\, dt\right) \\ &\geq \frac{k_1}{4}(1-x)^2(1-y)^2(x \wedge y)\left(\frac{1}{x \wedge y}\int_0^{x \wedge y} a(t)\, dt\right)^{-2} \\ &\geq \frac{k_1}{4}(1-x)^2(1-y)^2(x \wedge y)^3 \\ &\equiv g_0(x, y). \end{aligned}$$

The first inequality stems from $a(x) \geq k_1(1-x)^3$, the second is Jensen's inequality, while the third uses the nonnegativity of a and the volume constraint (2.3). As

$$\int_0^1 \int_0^1 g_0(x, y)\, dx\, dy = \frac{k_1}{720},$$

our result follows. $\square$

Our second application of (3.1) states that the eigenvalues depend continuously on the Green's function.

PROPOSITION 3.3. *If a_1 and a_2 each lie in ad_p for $p < 3$ then*

$$\left|\frac{1}{\lambda_1(a_1)} - \frac{1}{\lambda_1(a_2)}\right| \leq \|g(\cdot, \cdot; a_1) - g(\cdot, \cdot; a_2)\|.$$

Proof. Set $d(a_1, a_2) \equiv \|g(\cdot, \cdot; a_1) - g(\cdot, \cdot; a_2)\|$. Hölder's inequality provides

$$\langle G(a_2)\phi, \phi\rangle - d(a_1, a_2) \leq \langle G(a_1)\phi, \phi\rangle \leq \langle G(a_2)\phi, \phi\rangle + d(a_1, a_2),$$

when $\|\phi\| = 1$. Applying (3.1) throughout now gives

$$\frac{1}{\lambda_1(a_2)} - d(a_1, a_2) \leq \frac{1}{\lambda_1(a_1)} \leq \frac{1}{\lambda_1(a_2)} + d(a_1, a_2). \qquad \square$$

As preparation for our result on rearrangements we express (3.1) in a form reminiscent of that invoked by Alvino & Trombetti [1].

PROPOSITION 3.4. *If $a \in ad_p$ with $p < 3$ then*

$$\frac{1}{\lambda_1(a)} = \max_{\|\phi\|=1} \int_0^1 \frac{1}{a^2(x)}\left(\int_x^1 \sqrt{w(y)}\phi(y)\, dy\right)^2 dx.$$

Proof. This follows directly from

$$\langle G(a)\phi,\phi\rangle = \int_0^1\int_0^1 \sqrt{w(x)w(y)}\int_0^{x\wedge y}\frac{dt}{a^2(t)}\phi(x)\phi(y)\,dy\,dx$$
$$= \int_0^1 \sqrt{w(x)}\phi(x)\left(\int_0^x \sqrt{w(y)}\phi(y)\int_0^y \frac{dt}{a^2(t)}\,dy + \int_x^1 \sqrt{w(y)}\phi(y)\int_0^x \frac{dt}{a^2(t)}\,dy\right)dx.$$

Integrating by parts brings

$$\int_0^x \sqrt{w(y)}\phi(y)\int_0^y \frac{dt}{a^2(t)}\,dy = -\int_0^x \frac{dt}{a^2(t)}\int_x^1 \sqrt{w(t)}\phi(t)\,dt + \int_0^x \frac{1}{a^2(y)}\int_y^1 \sqrt{w(t)}\phi(t)\,dt\,dy,$$

so

$$\langle G(a)\phi,\phi\rangle = \int_0^1 \sqrt{w(x)}\phi(x)\int_0^x\int_y^1 \sqrt{w(t)}\phi(t)\,dt\frac{dy}{a^2(y)}\,dx.$$

Integrating this by parts, we arrive at the final form

$$\langle G(a)\phi,\phi\rangle = \int_0^1 \frac{1}{a^2(x)}\left(\int_x^1 \sqrt{w(t)}\phi(t)\,dt\right)^2 dx. \qquad \square$$

4. Increasing height via decreasing rearrangement. Expecting that the most efficient use of material will start from a large base and suffer a gradual diminution, we here show that replacing a by its decreasing rearrangement can but increase $\lambda_1(a)$. We follow a line of reasoning which, in our context, goes back to Krein [10] and Beesack & Schwarz [2]. The former considered the effect of the rearrangement of mass density while the latter addressed the rearrangement of a potential term. Our problem, with the design variable appearing in a nonlinear fashion in the highest order term and in a nonlocal fashion in the lowest order term, is considerably more cumbersome. Our contribution amounts to striking upon a variational characterization of $\lambda_1(a)$ which permits the application of the methods of [10] and [2].

Recall that the decreasing rearrangement of a nonnegative function, f, on $(0,1)$ is simply

$$f^*(x) \equiv \sup\{t>0 : \mu_f(t) > x\},$$

where

$$\mu_f(t) = |\{x\in(0,1) : f(x) > t\}|$$

is the measure of the set on which f exceeds t. The increasing rearrangement of f is simply $f_*(x) \equiv f^*(1-x)$. It is not difficult to show that

$$\int_0^1 f\,dx = \int_0^1 f^*\,dx = \int_0^1 f_*\,dx. \tag{4.1}$$

Regarding integrals of products, we recall the following propposition.

PROPOSITION 4.1. *Let f,ξ, and η be nonnegative functions, with ξ increasing and η decreasing. Then*

$$\int_0^1 f^*\xi\,dx \le \int_0^1 f\xi\,dx \quad \text{and} \quad \int_0^1 f_*\eta\,dx \le \int_0^1 f\eta\,dx. \tag{4.2}$$

Proof. These are both special cases of inequalities established in Pólya and Szegö [11, p. 153]. □

As final preparation we recall the increasing rearrangement of a certain composition.

PROPOSITION 4.2. *If ψ is decreasing on the range of f then $(\psi \circ f)_* = \psi \circ f^*$.*

Proof. This is a special case of Cox [3, Theorem 1]. □

PROPOSITION 4.3. *If $a \in ad_p$ and $p < 3$ then $\lambda_1(a) \le \lambda_1(a^*)$.*

Proof. Denote by v the eigenfunction of $G(a^*)$ corresponding to $\lambda_1(a^*)$. As previously remarked, v is nonnegative. Now

$$\begin{aligned}
\frac{1}{\lambda_1(a)} &\ge \int_0^1 \frac{1}{a^2(x)} \left(\int_x^1 \left(\int_y^1 a(t)\,dt \right)^{1/2} v(y)\,dy \right)^2 dx \\
&\ge \int_0^1 \frac{1}{a^2(x)} \left(\int_x^1 \left(\int_y^1 a^*(t)\,dt \right)^{1/2} v(y)\,dy \right)^2 dx \\
&\ge \int_0^1 \left(\frac{1}{a^2(x)} \right)_* \left(\int_x^1 \left(\int_y^1 a^*(t)\,dt \right)^{1/2} v(y)\,dy \right)^2 dx \\
&= \int_0^1 \frac{1}{(a^*)^2(x)} \left(\int_x^1 \left(\int_y^1 a^*(t)\,dt \right)^{1/2} v(y)\,dy \right)^2 dx \\
&= \frac{1}{\lambda_1(a^*)}.
\end{aligned}$$

The first inequality is a direct consequence of Proposition 3.4. The second inequality comes from the first in (4.2) with ξ being the characteristic function of $(x, 1)$. The third inequality follows from the second in (4.2) with

$$\eta(x) = \left(\int_x^1 \left(\int_y^1 a^*(t)\,dt \right)^{1/2} v(y)\,dy \right)^2 .$$

We remark that the nonnegativity of v leads to the nonincreasing of η. The first equality is a consequence of Proposition 4.2 with $\psi(t) = t^{-2}$. The final equality follows directly from the definition of v. □

5. Existence of a tallest column. Let us denote by ad_p^1 the collection of $a \in ad_p$ obeying the integral constraint (2.3). For $p < 3$ it follows from Proposition 3.2 that λ_1 is bounded on ad_p^1 and so

$$\lambda_1^{(p)} \equiv \sup_{a \in ad_p^1} \lambda_1(a)$$

is finite. That this sup is attained will follow directly from the following proposition.

PROPOSITION 5.1 (Helly's selection theorem, [13, p. 167]). *If $\{f_n\}$ is a sequence of nonnegative nonincreasing functions on $[0, 1]$ then there exists a subsequence $\{f_{n_k}\}$ and a function f such that $f_{n_k}(x) \to f(x)$ as $k \to \infty$ for each $x \in [0, 1]$.*

PROPOSITION 5.2. *If $p < 3$ then $a \mapsto \lambda_1(a)$ attains its maximum on $a \in ad_p^1$.*

Proof. As $\lambda_1^{(p)} < \infty$ there exists a maximizing sequence $\{a_n\} \subset ad_p^1$ for which $\lambda_1(a_n) \to \lambda_1^{(p)}$. By (4.1) and Proposition 4.3 we may assume that each a_n is nonincreasing and hence, by Helly's selection theorem, that there exists an $\hat{a}$ and a subsequence (that we neglect to relabel) such that $a_n \to \hat{a}$ pointwise. It follows by the dominated convergence theorem (Rudin [13, Theorem 11.32]) that

$$\int_x^1 a_n(t)\,dt \to \int_x^1 \hat{a}(t)\,dt \quad \text{and} \quad \int_0^{x\wedge y} \frac{dt}{a_n^2(t)} \to \int_0^{x\wedge y} \frac{dt}{\hat{a}^2(t)}$$

for each x and y. In particular,

$$\int_0^1 a_n\,dx \to \int_0^1 \hat{a}\,dx \quad \text{and} \quad g(x,y;a_n) \to g(x,y;\hat{a}).$$

By the dominated convergence theorem it follows that $g(\cdot,\cdot,a_n) \to g(\cdot,\cdot,\hat{a})$ in $L^2((0,1)\times(0,1))$. This implies, via Proposition 3.3, that $\lambda_1(a_n) \to \lambda_1(\hat{a})$. But, by construction, $\lambda_1(a_n) \to \lambda_1^{(p)}$, and so $\lambda_1(\hat{a}) = \lambda_1^{(p)}$. □

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