CALCULUS OF VARIATIONS

1. INTRODUCTION

Functional: Mapping of some function space into real line.

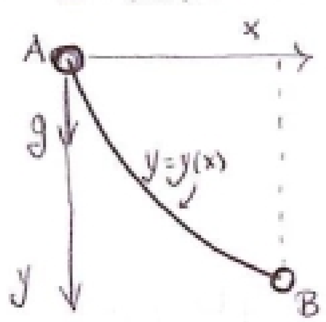
eg:
$$S[y] = \int_{x_0}^{x_1} \sqrt{1+[y'(x)]^2}\, dx \quad \text{(arclength)} \tag{1}$$

$$T(f) = \max_{0 \le x \le 1} f(x) \tag{2}$$

"**Find** function for which functional is at extremum (perhaps subject to additional constraints)".

Classic problems

I Brachistochrone (John Bernoulli, 1696)



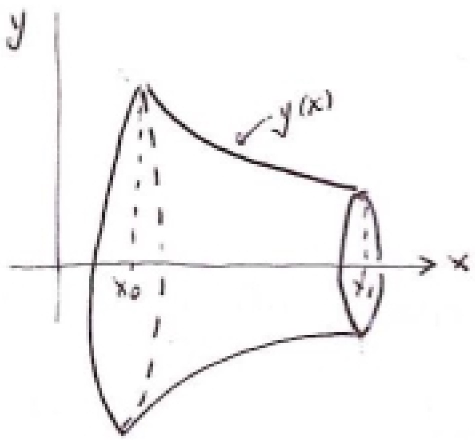
Travel time

$$T = \int_A^B \frac{ds}{v}, \quad \left(\text{speed } v = \frac{ds}{dt}\right)$$
$$= \int_A^B \frac{\sqrt{1+[y'(x)]^2}}{v}\, dx \tag{3}$$

KE $\frac{1}{2}mv^2 = mgy \Rightarrow v(x) = \sqrt{2g\,y(x)}$

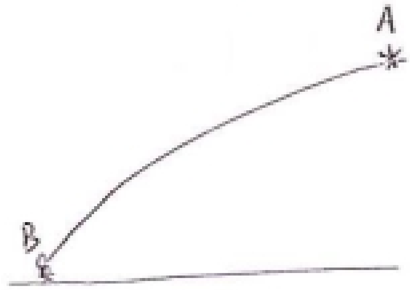
$$\sqrt{2g}\, T[y] = \int_0^{x_B} \frac{\sqrt{1+[y'(x)]^2}}{\sqrt{y(x)}}\, dx$$; "Minimize T wrt y"

II Plateau's problem (minimal surface of revolution)



$$\min \int_{x_0}^{x_1} y \sqrt{1 + [y'(x)]^2}\, dx \tag{4}$$

III. <u>Fermat's principle</u> (2-dim, inhomogeneous refractive in $c(x,y)$)



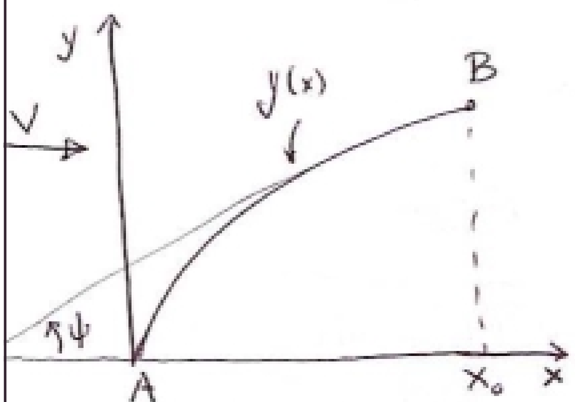
$$dt = \frac{1}{c} ds$$

$$\min t = \int \frac{ds}{c} = \int_A^B \frac{\sqrt{1+[y'(x)]^2}}{c(x,y)}\, dx \tag{5}$$

(principle of least time)

IV <u>Newton 1686</u> (minimize drag on body of revolution)

Postulated: $D = 2\pi\rho V^2 \int y \sin^2\psi\, dy$



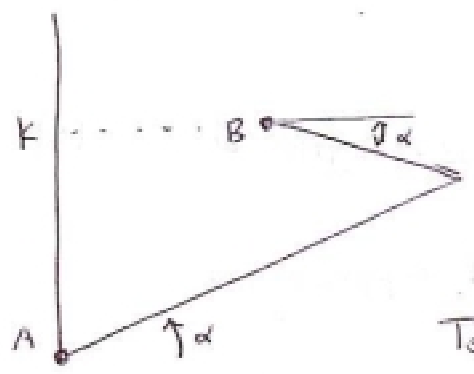
ρ: air density

V: speed

$\tan\psi = y'$ (slope)

$$\sin^2\psi\, dy = \sin^2\psi \frac{dy}{dx} dx = \sin^2\psi \tan\psi\, dx$$
$$= \tan^3\psi \cos^2\psi\, dx = \frac{\tan^3\psi}{\sec^2\psi} dx$$
$$= \frac{\tan^3\psi}{1+\tan^2\psi} dx = \frac{y'^3}{1+y'^2} dx$$

i.e. $$D = 2\pi\rho V^2 \int_0^{x_0} \frac{y y'^3}{1+y'^2} dx \tag{6}$$



Can make integral as small as we want:

$$\int_0^k y\sin^2\psi\, dx = \sin^2\alpha \int_0^k y\,dy = \frac{1}{2}k^2\sin^2\alpha$$

By making α smaller, $D \to 0$

To avoid this, add restriction $0 \le \psi \le \frac{\pi}{2}$.

V <u>Isoperimetric problem</u>: $\max \int y(x)\,dx$

subject to $\int \sqrt{1+[y']^2}\,dx = \text{constant}$ $\left(\int ds = l, \text{length}\right)$ (7)

(Arc of fixed length $\Rightarrow$ max area for circle
surf. of fixed area $\Rightarrow$ max volume for sphere)

VI: <u>Column</u> of fixed volume: maximize load strength.

VII <u>Geodesics</u>: shortest distance between two points on surface

VIII <u>Hamilton's Principle</u> of least action

Generally:

extremize $\int F(x,y,y')\,dx$ (8a)

subject to $\int G(x,y,y')\,dx = \text{constant}$ (8b)

Conservation laws and Differential equations

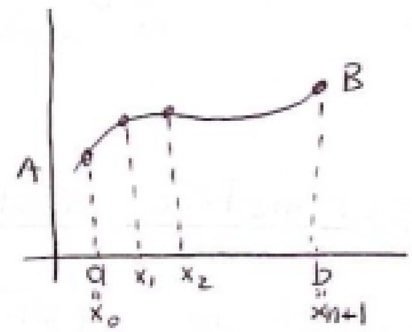
Direct $\longleftrightarrow$ Indirect

DE Extrema

2. Numerical Methods

$$I[y] = \int_a^b F(x, y(x), y'(x))\,dx\,; \quad y(a) = A\,,\; y(b) = B \qquad (1)$$

Classical approaches



1. Finite differences

$$x_{i+1} - x_i = h$$

Replace curve* $y(x)$ by polygonal approximation with vertices
$(\underset{a}{x_0}, A), (x_1, y_1), \ldots, (x_n, y_n := y(x_n)), (\overset{b}{x_{n+1}}, B)$ (where $y_i := y(x_i)$)

Approximate $I[y]$ by

$$I_n := I(y_1, \ldots, y_n) = \sum_{i=1}^{n+1} F\left(x_i, y_i, \frac{y_i - y_{i-1}}{h}\right) h$$

$$\therefore \frac{\partial I_n}{\partial y_i} = 0$$

(Sequence $\{y_i\}$ is called a "minimizing sequence")

Questions: (i) Is $\lim_{n \to \infty} I_n = \min_y I[y]$

(ii) Does polygonal line converge to the minimizer?

2. General minimizing sequences; the Rayleigh-Ritz method.

Assume $I[y]$ possesses a glb; that is,

* the length of a curve is defined as the limit of the sequence of lengths of inscribed polygonal lines. If limit exists, curve is called rectifiable.

assume $\inf_{y \in \mathcal{A}} I[y] = d > -\infty$ for $y \in \mathcal{A}$, some class.

Then $\exists$ sequences $\phi_1, \phi_2, \ldots$ of admissible function such that $\lim_{n\to\infty} I[\phi_n] = d$ while $I[\phi_i] \geq d$ for all $\Phi_i \in \mathcal{A}$. $\{\Phi_n\}$ are called minimizing sequences.

<u>Questions</u> (i) does $\lim_{n\to\infty} \phi_n$ exist?

(ii) is $\lim_{n\to\infty} I[\phi_n] = \min I[y]$?

{<u>Ritz</u>: (to minimize $I[y]$)} Choose a complete set of admissible functions $u_1, u_2, u_3, \ldots$ Define a sequence $\{y_n\}$ ($\longrightarrow$ minimizing sequence) by

$$y_n = \sum_{i=1}^{n} a_i u_i$$

Consider $I[y_n]$; it is now a function of the $\{\alpha_i\}_1^n$, the n variables $\alpha_i, i=1, \ldots$ Minimize $I[y]$ as a function of that is $\partial I / \partial \alpha_i = 0$. Clearly $\min I[y] \leq I[y_n]$

<u>Ex</u> $\min I[y] = \int_0^1 (x^3 y''^2 + 100xy - 20xy)\,dx$; $y(1) = y'(1) = 0$

Complete set: $\{x^r\}$ (Weierstrass polynomial approximation theorem: polynomials are complete)

$$y_n(x) = (x-1)^2 (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)$$

Take $y_1(x) = (x-1)^2 (a_0 + a_1 x)$

$y_1''(x) = 6a_1 x + 2a_0 - 4a_1$

Then $\int_0^1 \left[x^3 (6a_1 x + 2a_0 - 4a_1)^2 + 10x(x-1)^4 (a_0 + a_1 x)^2 - 2x(x-1)(a_0 + a_1 x) \right] dx$

$$\therefore \quad I = I(a_0, a_1)$$

$$\partial I/\partial a_0 = \partial I/\partial a_1 = 0 \ : \ a_0 = .124, \ a_1 = .218$$

Minimizer : $I_1 = I[y_1]$: $\quad y_1 = (x-1)^2 (.124 + .218\,x)$

— . —

3. Difficulties:

(i) a minimizing sequence need not converge

(ii) even if a ms converges, its limit may be inadmissible

(iii) It is possible to have a seq. of admissible function converging to the lub or glb of all possible values and yet have the value of the integral for the limit fn. itself be different from the lub or glb

Example: (i) $I = \iint_D (\phi_x^2 + \phi_y^2)\, dx\, dy = \iint_D \left(\phi_r^2 + \frac{1}{r^2} \phi_\theta^2 \right) r\, dr\, d\theta$

Set $\phi_n = \begin{cases} c_n \operatorname{Log} a_n, & 0 \le r \le a_n^2 \\ c_n \operatorname{Log}(r/a_n) & a_n^2 \le r \le a_n \\ 0 & a_n \le r \le 1 \end{cases}$

where the c_n, a_n are constant

Then $I[\phi_n] = 2\pi c_n^2 \int_{a_n^2}^{a_n} \frac{1}{r^2}\, r\, dr = -2\pi c_n^2 \operatorname{Log} a_n$

Now choose $a_n = e^{-n}$ and $c_n = -n^{-2/3}$. Then

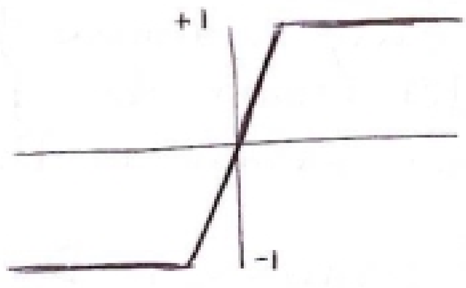
$$I[\phi_n] = 2\pi / n^{1/3} \longrightarrow 0 \quad \text{as } n \to \infty$$

But $\phi_n(0) = n^{1/3} \longrightarrow \infty$ as $n \to \infty$.

Example (ii) min $I = \int_{-1}^{1} x^2 [u'(x)]^2 dx$; $u(-1) = -1$, $u(1) = 1$

$I \geq 0$. Set

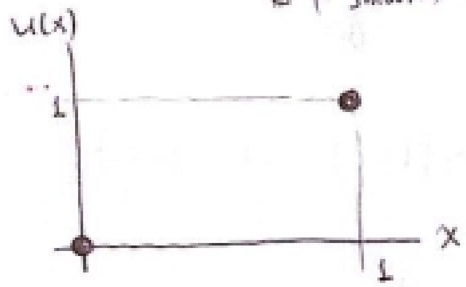
$$u_n(x) = \begin{cases} -1, & -1 \leq x \leq -\frac{1}{n} \\ nx, & -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 1 & \frac{1}{n} \leq x \leq 1 \end{cases}$$



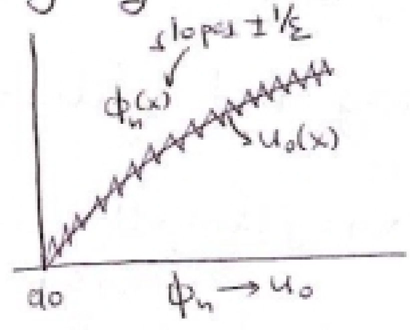
Then $I(u_n) = \int_{-1/n}^{1/n} n^2 x^2 dx = \frac{2}{3n} \to 0$ as $n \to \infty$

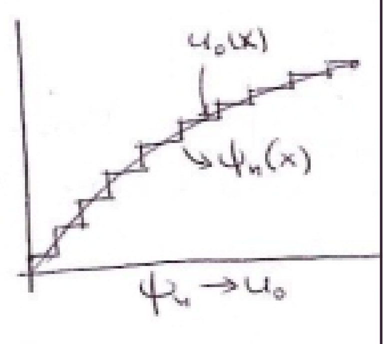
But $u_n \to u = \begin{cases} +1, & x > 0 \\ -1, & x < 0 \end{cases}$ i.e. not an admissible function (i.e. not continuous, piecewise smooth)

Example (iii) (piecewise smooth) min/max $I = \int_0^1 \frac{1}{1 + [u'(x)]^2} dx$; $u(0) = 0$, $u(1) = 1$



Clearly glb $I = 0$, lub $I = 1$





$I(\phi_n) \to 0 \neq I(u_0) \neq 1 \leftarrow I(\psi_n)$

3. Basic Analysis definitions

1. Theorem (Weierstrass)

Every continuous function defined on a closed, bounded domain D of n-dimensional space possesses a min and max in D.

▲ f continuous in $D \Rightarrow f$ bounded for all $x \in D$

Let $d = \inf_{x \in D} [f(x)]$. Then $\exists$ sequence $\{x_n\}$ of points in D such that $f(x_n) \to d$

and D ~~closed + bdd~~ compact $\Rightarrow$ we can select a convergent subsequence $\{x_n'\}$ of seq. $\{x_n\}$

(continuity of f) &

i.e. $\lim\limits_{n\to\infty} x_n' = x_0 \in D$, $\lim\limits_{n\to\infty} f(x_n') = f(x_0) = d$

$\therefore$ f continuous, D compact : $f(\lim x_n') = \lim f(x_n') = d$

i.e. f achieves its minimum ▼

If we knew merely that $f(\lim x_n') \leq \lim f(x_n')$ the proof still works because

$$f(x_0) = f(\lim x_n') \leq \lim f(x_n') = d$$
$$\Rightarrow f(x_0) \leq d$$

But by definition of d we have $f(x) \geq d$ for all $x \in D$.

$$\Rightarrow \{ f(x_0) \geq d \ \wedge \ f(x_0) \leq d \} \Rightarrow f(x_0) = d$$

<u>Defn</u>: sequence $\{a_n\}$; suppose U satisfies

(i) Ultimately all terms of $\{a_n\}$ lie to the left of U (i.e. $\forall\, \varepsilon > 0 \ \exists\, N > 0 \ \ni \{n > N \Rightarrow a_n < U + \varepsilon\}$

(ii) Infinitely many terms lie to the right of $U - \varepsilon$

(i.e. given $\varepsilon > 0$ and given $m > 0$ $\exists$ integer $n > m$

$\exists\ a_n > U - \varepsilon$

U is called limit superior

$$U = \limsup_{n\to\infty} a_n = \overline{\lim_{n\to\infty}}\, a_n$$

Ex (i) $a_n = \dfrac{(-1)^n}{1+\frac{1}{n}}$, $\underline{\lim} = -1$, $\overline{\lim} = +1$

(ii) $a_n = (-1)^n$, $\underline{\lim} = -1$, $\overline{\lim} = +1$

(iii) $a_n = (-1)^n n$, $\underline{\lim} = -\infty$, $\overline{\lim} = +\infty$

(iv) $a_n = n^2 \sin^2\left(\frac{1}{2} n\pi\right)$, $\underline{\lim} = 0$, $\overline{\lim} = +\infty$

2. Defn: A lower semi-continuous function is defined as a function $f \ni f(\lim x_n) = \underline{\lim}\, f(x_n)$ for every convergent sequence $\{x_n\}$ in the domain of f.

Theorem: Every lower (upper) semi-continuous function defined on a closed and bounded set D possesses a min (max) in D.
($f(x)$ has a max at $x = x_0$ if $f(x) < f(x_0)$ for all x in a deleted neighborhood of x_0).

* S: function space, $\underline{x} = (x_1, \ldots, x_n)$

Functional: mapping $S \to \mathbb{R}$

Linear space: $x+y = y+x$, $(\alpha+\beta)x = \alpha x + \beta x$

3. Linear space $\mathcal{M}$ is said to be normed if to each $x \in \mathcal{M}$ we can assign a non-negative number $\|x\|$ called the norm of x, $\ni$:

$$\begin{cases} \|x\| = 0 \Leftrightarrow x = 0 \\ \|\alpha x\| = |\alpha| \|x\| \\ \|x+y\| \leq \|x\| + \|y\| \end{cases}$$

Ex: (i) Space $C[a,b]$ of all continuous function on AB.
Define norm of an element $y(x) \in C[a,b]$ as

$$\|y\|_0 = \max_{a \leq x \leq b} |y(x)|$$

(ii) Space $C^1[a,b]$ ($\equiv \mathcal{D}_1[a,b]$ in Gelfand & Fomin) of all functions which are continuous and have continuous first derivatives on $[a,b]$.

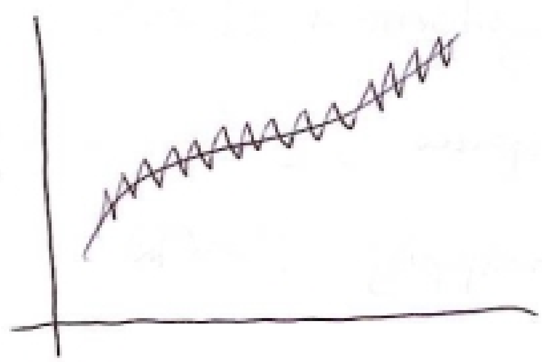
$$\|y\|_1 = \max_{x \in [a,b]} |y(x)| + \max_{x \in [a,b]} |y'(x)|$$

(iii) $C^n[a,b]$; $\|y\|_n = \sum_{i=0}^{n} \max_{x \in [a,b]} |y^{(i)}(x)|$

$$\begin{cases} \text{I}' \; C_p[a,b] \\ \text{II}' \; C_p^1[a,b] \\ \text{III}' \; C_p^{(n)}[a,b] \end{cases}$$

metric $\rho(u,v) = \|u-v\|_p$

Ex.



Close in C
but not in C'