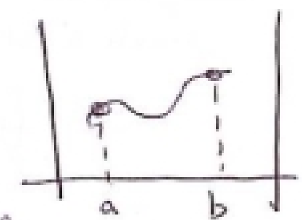
4. Functionals

4.1. Definition: weak variation ($\delta y, \delta y'$ small)
strong " δy small

Definition: The functional is said to be continuous at the point $y_0 \in M$ if $\forall \varepsilon > 0 \; \exists \delta > 0$
$\ni |I(y) - I(y_0)| < \varepsilon$ provided that $\|y - y_0\| < \delta$.

* A functional can be continuous in some space and discontinuous in another.

Ex: Arclength $\quad S = \int_a^b \sqrt{1 + [y'(x)]^2}\, dx \equiv I[y]$



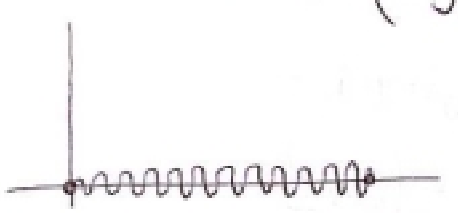
① $I[y]$ is continuous in $C^1[a,b]$, because clearly
$\|y_1 - y_2\| < \delta \Rightarrow |y_1'(x) - y_2'(x)| < \delta \Rightarrow |I[y_1] - I[y_2]| < \varepsilon$

② $I[y]$ is NOT continuous in the $\|\ \|_0$ norm, even in C^1.

Take $y(x) = a \sin(r\pi x)$ in $x \in [0,1]$; then

$$S = \int_0^1 \sqrt{1 + r^2\pi^2 a^2 \cos^2(r\pi x)}\, dx > r\pi a \int_0^1 |\cos r\pi x|\, dx$$

$$= r\pi a \left(2r \int_0^{1/2r} \cos \pi r x\, dx \right) = r\pi a \cdot 2r \cdot \frac{1}{\pi r} = 2ar$$



Take $a = \frac{1}{n}$, $r = n^2$. Then

$I > 2n \Rightarrow I \to \infty$ as $n \to \infty$
but $|y| \le \frac{1}{n} \to 0$. $\left(y' = ar\pi \cos r\pi x = n\pi \cos n^2 \pi x \atop \|y\|_1 = n\pi + \frac{1}{n} \to \infty \right)$

$\therefore \quad I[\lim \phi_n] \le \underline{\lim}\, I[\phi_n]$, lower semicontinuous
$\qquad \overset{\|}{0}$

<u>Theorem</u>: Every lower (upper) semicontinuous functional defined on a compact* function space possesses a min (max) in that space [proved using equicontinuity and Arzela-Ascoli: look in Courant-Friedrichs].

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* <u>Definition</u>: A set of functions $\{f_n(x)\}$ in a space M is said to be compact if every sequence of functions in the set contains a subsequence which converges to some function in M.

$$f(x_1, \ldots, x_n) \;;\; df = f'(x)\,dx$$

$f(x)$ $\left\{ \begin{array}{l} \text{necessary} \\ \text{sufficient} \end{array} \right.$ conditions for min at $x = x_0$ are $\begin{array}{l} f'(x_0)=0,\ f''(x_0)\geq 0 \\ f''(x_0)>0 \end{array}$

Ordinary functions: we say that $f(x)$ is differentiable at x, $f(x+h)-f(x) = \underbrace{A\cdot h}_{\text{differential}} + o(|h|)$ where $A\cdot h = \sum_{i=1}^{n} A_i h_i$

$$\lim_{\varepsilon\to 0}\left[\frac{f(x+\varepsilon\eta)-f(x)}{\varepsilon}\right] = A\cdot\eta \quad , \text{ derivative: } \underline{A} = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

$$A = \nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

$$f(x+h) - f(x) = \underbrace{A\cdot h}_{\text{linear form}} + (\text{quadratic form})$$

<u>Ex</u>: $f(x+h, y+k) - f(x,y) = f_x(x,y)\,h + f_y(x,y)\,k +$

$$+\frac{1}{2!}\left[f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2\right]$$

$$+\frac{1}{3!}\left[f_{xxx}h^3 + 3f_{xxy}h^2k + 3f_{xyy}hk^2 + f_{yyy}k^3\right] + \ldots$$

4.2 Linear functionals

<u>Defn</u>: $I(\alpha y_1 + \beta y_2) = \alpha I(y_1) + \beta I(y_2)$: <u>linear functional</u>

Ex. $I(y) = \int_a^b f(x) y(x)\,dx$

$I(y) = y(x_0)$

$I(y) = \int_a^b [\alpha_0(x) y(x) + \alpha_1(x) y'(x) + \cdots + \alpha_n(x) y^{(n)}(x)]\,dx$

<u>Defn</u>: functional $B(x,y)$ is bilinear if it is linear in $y(x)$ for fixed x (y).

i.e. $B(\alpha x + \beta y, z) = \alpha B(x,z) + \beta B(y,z)$

$B(x, \alpha y + \beta z) = \alpha B(x,y) + \beta B(x,z)$

If we set $y = x$ we get a <u>quadratic functional</u> $A(x) = B(x,x)$.

If $A(x) > 0$ for $x \neq 0$, then <u>positive definite</u>

Ex. (i) $B(x,y) = \sum_{i,j=1}^{n} b_{ij} x_i y_j$

$A(x) = B(x,x) = \sum_{ij=1}^{n} b_{ij} x_i x_j$

(ii) $B(x,y) = \int_a^b \alpha(t) x(t) y(t)\,dt$

$A(x) = \int_a^b \alpha(t) x^2(t)\,dt$

(iii) $B(x,y) = \int_a^b \int_a^b k(s,t) x(t) y(t)\,ds\,dt$

4.3 Defn: <u>$J(y)$ differentiable functional</u>

Suppose $\underbrace{J(y+h) - J(y)}_{\Delta J} = \phi(h) + \varepsilon \|h\|$

more precisely, $\phi(y,h)$

where $\varepsilon \to 0$ as $\|h\| \to 0$ and where $\phi(h)$ is a linear functional. Then the functional $J(y)$ is said to be differentiable at y and the linear functional $\phi(h)$ is

Called the first variation (or differential) of J and denoted by $\delta J[h]$

<u>Riesz representation theorem</u>. A linear functional $\phi(h)$ can be represented uniquely as the inner product

$$\Phi(h) = (h, u).$$

Define u as the derivative

<u>Def</u>: $$\Delta J = J(y+h) - J(y) = \underbrace{\overset{\to \delta J}{\Phi_1(h)}}_{\text{linear in } h} + \underbrace{\overset{\to \delta^2 J}{\Phi_2(h)}}_{\text{quad in } h} + \overset{\text{i.e. } o(\|h\|^2)}{\varepsilon \|h\|^2}$$

<u>Ex</u>: $J(y) = \int_a^b F(x, y(x), y'(x))\,dx$

$$\Delta J = J(y+h) - J(y) = \int_a^b F(x, y+h, y'+h')\,dx - \int_a^b F(x,y,y')\,dx$$

$$= \int_a^b \left[F_y(x,y,y')\,h + F_{y'}(x,y,y')\,h' \right] dx +$$

$$+ \frac{1}{2}\int_a^b \left[F_{yy}\,h^2 + 2F_{yy'}\,h h' + F_{y'y'}\,h'^2 \right] dx + \cdots$$

$$\delta J := \int_a^b (F_y h + F_{y'} h')\,dx$$

$$\delta^2 J := \frac{1}{2}\int_a^b \left[F_{yy} h^2 + 2F_{yy'} h h' + F_{y'y'} h'^2 \right] dx$$

<u>Ex</u>: $J(y) = \int_a^b F(x,y,y')$, $y(a) = A$, $y(b) = B$; $h(a) = h(b) = 0$

$$\delta J = \int_a^b [F_y h + F_{y'} h']\,dx = \int_a^b \left[F_y h - \frac{d}{dx} F_{y'} h \right] dx \Rightarrow$$

$$\left(\int_a^b F_{y'} h'\,dx = \underbrace{F_{y'} h \Big|_a^b}_{=0 \text{ from BC on } h} - \int_a^b h \frac{dF_{y'}}{dx}\,dx \right)$$

So $\Delta J = J(y+h) - J(y) = \underbrace{\Phi(h)}_{\delta J} + o(\|h\|)$

$$\delta J = \lim_{\varepsilon \to 0} \left[\frac{J(y+\varepsilon\eta) - J(y)}{\varepsilon} \right] = \left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} \cdot \eta$$

<u>Ex</u>: $J(y) = \int_a^b u^2 dx, \quad \frac{\delta J}{\delta y} = 2y$

4.4. <u>EXTREMA OF FUNCTIONALS</u> (max or min)

Functional $J(y)$ has a relative extremum for $y = y_0$ if $J(y) - J(y_0)$ does not change sign in some neighborhood $(\|\cdot\|_i)$ of y_0 almost always (in $C[a,b)$ or $C'[a,b]$)

1. <u>Defn</u>: $J(y)$ has extremum for $y = \hat{y}$ if $\exists\ \varepsilon > 0 \ni$ $J(y) - J(\hat{y})$ has the same sign for all y in domain of definition of the functional which satisfy

$\|y - \hat{y}\|_1 < \varepsilon$ (weak variation $\left.\begin{matrix} y-\hat{y} \\ y'-\hat{y}' \end{matrix}\right\}$ small)

or

$\|y - \hat{y}\|_0 < \varepsilon$ (strong variation : $y - \hat{y}$ small)

1. <u>Theorem</u>: A necessary condition for the differentiable functional J to have an extremum at $y = \hat{y}$ is that $\delta J = 0$ at $\hat{y}$.

Suppose $\delta J \neq 0$; $\Delta J = J(y+h) - J(y) = \delta J(h) + \varepsilon \|h\|$

$$\Rightarrow \delta J = \int_a^b \left(F_y - \frac{d}{dx} F_{y'} \right) h \, dx = \left(F_y - \frac{d}{dx} F_{y'}, h \right)$$

~~Sim~~ Functional derivative:

$$\frac{\delta J}{\delta y} := F_y - \frac{d}{dx} F_{y'}$$

Ex (1) $J(y) = y(x_0)$

$$J(y+h) - J(y) = y(x_0) + h(x_0) - y(x_0) = h(x_0) = \delta J$$

$$\Rightarrow \delta J = \int \delta(x - x_0) \, h(x) \, dx$$

i.e. $$\frac{\delta J}{\delta y} = \delta(x - x_0)$$

4. Ex (2) $J(y) = \int_a^b \int_a^b K(x,y) y(x) y(\xi) \, dx \, d\xi$; $K(x,\xi) = K(\xi, x)$

$$\Delta J \equiv J(y+h) - J(y) = \iint K \left[y(x) h(\xi) + h(x) y(\xi) \right] dx \, d\xi + O(h^2)$$

$$= 2 \iint K(x,y) \, y(\xi) \, h(x) \, d\xi \, dx \quad \text{(because } K \text{ symmetric)}$$

$$= \int_a^b \left\{ 2 \int_a^b K(x,\xi) \, y(\xi) \, d\xi \right\} h(x) \, dx$$

$$\Rightarrow \frac{\delta J}{\delta y} = 2 \int_a^b K(x,\xi) \, y(\xi) \, d\xi$$

Take some admissible $h_0 \ni \delta J(h_0) \neq 0$. For arbitrary $\varepsilon > 0$, $J(\hat{y} + \varepsilon h_0) - J(\hat{y}) = \delta J(\varepsilon h_0) + \varepsilon \|h_0\| = \varepsilon \delta J(h_0) + \varepsilon \|h_0\|$.

$$J(\hat{y} - \varepsilon h_0) - J(\hat{y}) = \delta J(-\varepsilon h_0) = \underbrace{-\varepsilon \delta J(h_0)}_{\text{by linearity of } 1^{st} \text{ variation}} + \varepsilon \|h\|$$

i.e. $J(\hat{y})$ cannot be an extremum

DEF (2) J is stationary at $\hat{y}$ if $\delta J = 0$ at $\hat{y}$.

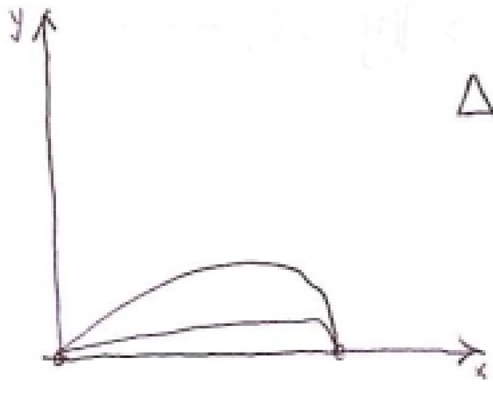
THM (2) Necessary condition for $J(y)$ to have $\underset{(\max)}{\min}$ at $\hat{y}$ is $\delta^2 J \underset{(\leq)}{\geq} 0$ at $\hat{y}$.

$$\Delta J = J(y+h) - J(y) = \underbrace{\delta J(h)}_{=0} + \underbrace{\delta^2 J(h)}_{\text{quadratic (i.e. of constant sign)}} + \underbrace{\varepsilon}_{0 \text{ as } \|h\| \to 0} \|h\|^2$$

NOTE: these are NOT sufficient conditions.

Ex (3) $I(y) = \int_0^1 (y'^2 + y'^3)\,dx$; $y(0) = 0$, $y(1) = 0$

$y \in C'[a,b]$



$$\Delta I = I(y+h) - I(y) = \underbrace{\int_0^1 (2y'h' + 3y'^2 h')\,dx}_{\delta I} + \underbrace{\int_0^1 (h'^2 + 3y'h'^2)\,dx}_{\delta^2 I} + \int_0^1 h'^3\,dx$$

Clearly for $y \equiv 0$ we have $\delta I = 0$ and $\delta^2 I > 0$

but consider $y_\phi(x) = \left(1/\tan\phi\right)^{3/2} \left(x \tan\phi - \tan\phi x\right)$

for which $\|y_\phi\| \to 0$ as $\phi \to \pi/2$ but $I(y_\phi) \to -\infty$.

$\therefore$ $y \equiv 0$ is <u>not</u> a strong relative min $(I(0)=0)$

But $y=0$ does provide a weak relative minimum because

$$\|y\|_1 < \varepsilon \Rightarrow |y'| < \varepsilon \Rightarrow I \leq \int_0^1 (\varepsilon^2 + \varepsilon^3)\,dx < 2\varepsilon^2$$

% Let $\int_0^1 u'^2 dx = \varepsilon^2 c > 0$, $(0 < c \leq 1)$

then $\int_0^1 |u'^3|\,dx \leq \varepsilon^3 c$ and

$$\delta I = \int_0^1 (u'^2 + u'^3)\,dx \geqslant \underbrace{\int_0^1 u'^2 dx}_{\text{strictly positive}} - \int_0^1 |u'|^3 dx \geqslant \varepsilon^2 c - \varepsilon^3 c = \varepsilon^2 c(1-\varepsilon) > 0$$

i.e. $0 < \varepsilon^2 c(1-\varepsilon) \leq I(y_\varepsilon) \leq 2\varepsilon^2$, i.e. $I(y_\varepsilon) \xrightarrow{\varepsilon \to 0^+} 0^+$.

or $I(y) - I(0) > 0$ for all $y_\varepsilon \ni \|y_\varepsilon - 0\|_1 < \varepsilon$.