5. FUNDAMENTAL LEMMA OF THE CALCULUS OF VARIATIONS

5.1, Preliminaries

<u>Lemma I</u>: If $\alpha(x) \in C[a,b]$ and if $\int_a^b \alpha(x) h(x) dx = 0$ for every function $h(x) \in C[a,b] \ni h(a) = h(b) = 0$ then $\alpha(x) \equiv 0$ in $[a,b]$.

Suppose $\alpha(x_0) > 0$ for some $x_0 \in [a,b]$. Then by continuity $\alpha(x) > 0$ for $x \in [x_0-\delta, x_0+\delta]$ for δ suff. small, >0.

Take
$$h(x) = \begin{cases} (x - x_0 + \delta)(x - x_0 - \delta), & \text{for } x \in [x_0-\delta, x_0+\delta] \\ 0 & \text{otherwise} \end{cases}$$
for which $\int_a^b \alpha(x) h(x) dx > 0$, QED (by contradiction)

<u>Corollary</u>: same result if we replace C by $C^{(n)}$.

<u>Lemma II</u>: if $\alpha(x) \in C[a,b]$ and if $\int_a^b \alpha(x) h'(x) dx = 0$ $\forall\, h \in C'[a,b]$ with $h(a) = h(b) = 0$, then $\alpha(x) \equiv$ constant, x in $[a,b]$

$\left(\int_a^b \alpha h' = \alpha h \Big|_a^b - \int_a^b \alpha' h = 0 \ \forall h \Rightarrow \alpha' = 0 \text{ by I} \Rightarrow \alpha = \text{const} \right.$
does not work because don't know α' exists)

Choose $h(x) = \int_a^x [\alpha(\xi) - c] d\xi$ where $c = \frac{1}{b-a} \int_a^b \alpha(\xi) d\xi$

Then: $0 = \int_a^b \alpha(x) h'(x) dx = \int_a^b \alpha(x) h'(x) dx - c\underbrace{[h(b) - h(a)]}_{=0}$

$= \int_a^b [\alpha(x) - c] h'(x) dx = \int_a^b [\alpha(x) - c]^2 dx$
since $h' = \alpha(x) - c$

$\Rightarrow \alpha(x) \equiv c$ throughout $(\alpha \in C[a,b])$.

Lemma III: if $\alpha(x) \in C[a,b]$ and if $\int_a^b \alpha(x) h''(x)\,dx = 0$ for every $h \in C^{(2)}[a,b]$; $h(a) = h'(a) = h(b) = h'(b) = 0$

then $\alpha(x) = C_0 + C_1 x$ (C_0, C_1 constants)

◀ (similar, see G & F) ▶

Lemma IV If $\alpha(x), \beta(x) \in C[a,b]$ and if

$$\int_a^b [\alpha(x) h(x) + \beta(x) h'(x)]\,dx = 0$$

for every function $h(x) \in C'[a,b]$ s.t. $h(a) = h(b) = 0$ then $\beta(x)$ is differentiable and $\beta'(x) = \alpha(x)$, for $x \in [a,b]$

◀ (formal integration by parts gets the answer by assuming β differentiable : no good!)

Let $A(x) = \int_a^x \alpha(\xi)\,d\xi \Rightarrow \int_a^b \alpha h\,dx = \int_a^b A' h\,dx = \cancel{A h\big|_a^b}_{0} - \int_a^b A h'\,dx$

$$\Rightarrow \int_a^b (\alpha h + \beta h')\,dx = \int_a^b (-A + \beta) h'\,dx = 0$$

$$\Rightarrow (\text{Lemma II}) \quad -A + \beta = \text{constant} \Rightarrow \beta(x) = \text{constant} + \int_a^x \alpha(\xi)\,d\xi$$ ▶

$$\Rightarrow \beta' = \alpha(x)$$

5.2 Legendre's condition for weak extremum

Let $I(y) = \int_a^b F(x, y, y')\,dx$; $y(a) = A$, $y(b) = B$

($F \in C^2$ in all arguments)

$\delta I = \int_a^b (F_y h + F_{y'} h')\,dx$ i.e. $\frac{d}{dx} F_{y'} = F_y$ or

(Euler equation) $\quad \frac{d}{dx}F_{y'} - F_y = 0 \qquad (+\text{b.c.}) \qquad (1)$

is a necessary condition for extremum. All solutions are called extremals

Alternatively: $\quad F_{y'y'}\,y'' + F_{yy'}\,y' + F_{y'x} - F_y = 0 \qquad (2)$

(as a result y'' continuous where $F_{y'y'} \neq 0$).

<u>Theorem(1)(du Bois-Raymond)</u>: Suppose $y \in C'[a,b]$ and it satisfies the Euler equations with $F \in C^2$ in all arguments, then y has continuous 2^{nd} derivative wherever $F_{y'y'}(x,y,y') \neq 0$.

<u>2^{nd} variation</u>: $(h(a) = h(b) = 0)$

$$\delta^2 I = \frac{1}{2}\int_a^b \left(F_{yy}h^2 + \underbrace{2F_{yy'}hh'} + F_{y'y'}h'^2\right)dx$$

$$\int_a^b 2F_{yy'}hh'\,dx = \underbrace{h^2}_{=0} F_{yy'}\Big|_a^b - \int_a^b \left(\frac{d}{dx}F_{yy'}\right)h^2\,dx$$

$$\Rightarrow \quad \delta^2 I = \int_a^b \left(Ph'^2 + Qh^2\right)dx$$

where
$$P = \frac{1}{2}F_{y'y'} \qquad (3)$$
$$Q = \frac{1}{2}\left(F_{yy} - \frac{d}{dx}F_{yy'}\right) \qquad (4)$$

Apply theorem 4.4.2 which states that a necessary condition for $I(y)$ to have a minimum at $\hat{y}$ is that $\delta^2 I \geq 0$ at $\hat{y}$.

<u>Lemma (1)</u>: A necessary condition for

$$\delta^2 I = \int_a^b (Ph'^2 + Qh^2)\,dx, \quad h(a) = h(b) = 0,$$

$h(x) \in C'[a,b]$ to be non-negative is that $P \geq 0$ on $[a,b]$.

Suppose $P \geq 0$ doesn't hold. Then $P = -2\beta$ $(\beta > 0)$ at some point x_0. Then $\exists\, \delta > 0 \ni P(x) < -\beta$ in

$$a \leq x_0 - \delta < x < x_0 + \delta \leq b$$

Consider an admissible $h(x)$ given by

$$h(x) = \begin{cases} \sin^2 \frac{\pi(x-x_0)}{\delta} & \text{for } x_0 - \delta \leq x \leq x_0 + \delta \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\int_a^b (Ph'^2 + Qh^2)\,dx = \int_{x_0-\delta}^{x_0+\delta} P \frac{\pi^2}{\delta^2} \sin^2 \frac{\pi(x-x_0)}{\delta}\,dx + \int_{x_0-\delta}^{x_0+\delta} Q \sin^2 \frac{\pi(x-x_0)}{\delta}\,dx$$

$$< -\frac{2\beta\pi^2}{\delta} + 2M\delta, \quad \text{with } M = \max|Q|$$

which is negative for δ sufficiently small.

<u>THEOREM 1</u> (LEGENDRE): A necessary condition for the functional

$$I(y) = \int_a^b F(x,y,y')\,dx, \quad y(a) = A, \; y(b) = B$$

to have a weak relative minimum at $y = \hat{y}(x)$ is that the inequality

$$F_{y'y'} \geq 0 \qquad \text{(Legendre's condition)} \qquad (5)$$

be satisfied at every point of the curve $\hat{y}(x)$.
(Same for maximum $\rightarrow$ invert inequality).

<u>Ex.(1)</u>: $I(y) = \int_0^1 \underbrace{(y'^2 + 12xy)}_{F(x,y,y')}\, dx \quad ; \quad y(0) = 0,\ y(1) = 1$

$$F_y = 12x \ , \quad F_{y'} = 2y' \ ; \quad \frac{d}{dx} F_{y'} = 2y''$$

$$F_{y'y'} = 2 > 0$$

$$\frac{d}{dx} F_{y'} - F_y = 0 \quad \Rightarrow \quad \left.\begin{array}{l} 2y'' - 12x = 0 \\ y(0) = 0 \\ y(1) = 1 \end{array}\right\} \Rightarrow y = x^3$$

<u>Ex.(2)</u> $\quad I(y) = \int_a^b F(y, y')\, dx$

$$\frac{d}{dx} F_{y'} - F_y = 0$$

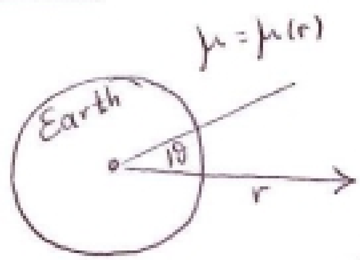
$$y'\left(F_{y'y'}\, y'' + F_{y'y}\, y' - F_y = 0\right) \qquad (\text{since } F_{y'x} = 0)$$

$$\Rightarrow \frac{d}{dx}\left(y' F_{y'} - F\right) = 0$$

$$\left(= (\underbrace{\cancel{\partial_x}}_{=0} + y'\partial_y + y''\partial_{y'})(y'F_{y'} - F) = y'^2 F_{yy'} - y' F_y + \cancel{y'' F_{y'}} + y'' y' F_{y'y'} - \cancel{y'' F_{y'}} \right)$$

$$\Rightarrow \quad y' F_{y'} - F = 0$$

Ex.(3) : Fermat's principle



$$\frac{ds}{dt} = c \Rightarrow dt = \frac{ds}{c} = \mu\, ds$$

minimize $\displaystyle\int \mu(r)\,ds = \int \mu(r)\sqrt{r'^2+r^2}\,d\vartheta \quad (*)_1$

$\displaystyle\int F(\vartheta, r, r')\,d\vartheta$
$\Downarrow$
$\left.\dfrac{d}{d\vartheta}F_{r'} - F_r = 0\right\}$

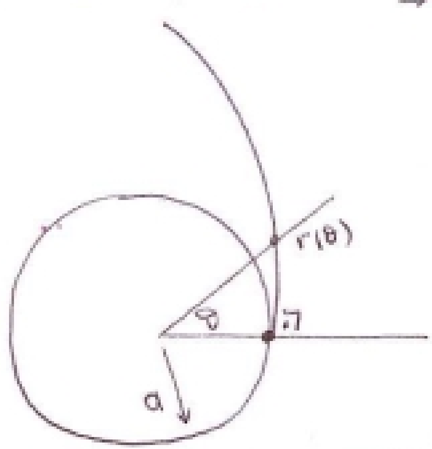
(assuming fixed BC- but leave unspecified)

$\Rightarrow$ (Ex.2): $r' F_{r'} - F = \text{const. } K$ (indep. of ϑ)

where $F = \mu(r)\sqrt{r'^2+r^2}$

$$\therefore \frac{r'^2\mu(r)}{\sqrt{r'^2+r^2}} - \mu(r)\sqrt{r'^2+r^2} = k$$

$$\Rightarrow \boxed{\frac{\mu(r)\, r^2}{\sqrt{r'^2+r^2}} = K} = a\,\mu(a)$$



If the light ray is at right angles to the radius vector at $\vartheta = 0$, $r = a$, we have $r' = 0$ at $r = a$ so

$$\boxed{K = a\mu(a)}$$

E.g. $\mu(r) = \sqrt{1+\frac{\sigma^2}{r^2}}$ Let $\mu(a) = \sqrt{1+\frac{\sigma^2}{a^2}} = \mu_0 > 1$

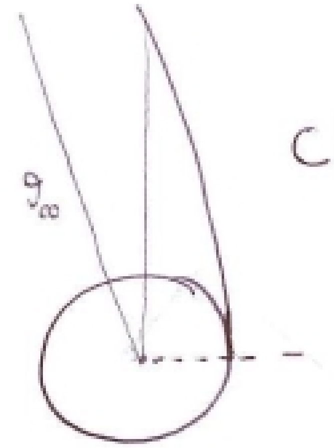
then, equ. for r' becomes

$$r'^2 = \frac{r^2(r^2-a^2)}{a^2\mu_0^2} \Rightarrow r(\vartheta) = a\sec(\vartheta/\mu_0)$$

$(*)_1$ $ds^2 = dx^2 + dy^2 = \left\{\left(\frac{dx}{d\vartheta}\right)^2 + \left(\frac{dy}{d\vartheta}\right)^2\right\} d\vartheta^2 = \{r'^2 + r^2\}\, d\vartheta^2$

$\left.\begin{array}{l} x = r(\vartheta)\cos\vartheta \\ y = r(\vartheta)\sin\vartheta \end{array}\right\}$ $\dfrac{dx}{d\vartheta} = r'\cos\vartheta - r\sin\vartheta$

$\dfrac{dy}{d\vartheta} = r'\sin\vartheta + r\cos\vartheta$



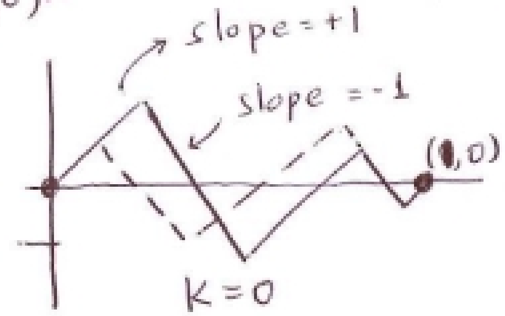
Clearly, $r = a$ when $\vartheta = 0$ while

$r \to \infty$ as $\vartheta \to \left(\mu_0 \frac{\pi}{2}\right)^-$

<u>Note</u>: for $\mu_0 \approx \frac{c_\infty}{c_\alpha} \sim 1.000294$ (Born & Wolf)

we find $\mu_0 \frac{\pi}{2} \simeq \frac{\pi}{2} + 1'35''$

<u>Ex.(4)</u> Minimize $\int_0^1 (y'^2 - 1)^2 \, dx$; $y(0) = 0$, $y(1) = k$



let $k = \tan\varphi$, $0 \le \varphi \le \frac{\pi}{2}$

Now, $I(y) \geq 0$; lower bound (i.e. 0) is not attained for any function $y \in C^1[0,1]$ (except for $\varphi = \pi/4$, $k = 1$; see below)

However, can construct infinitely many curves that have only piecewise continuous derivatives

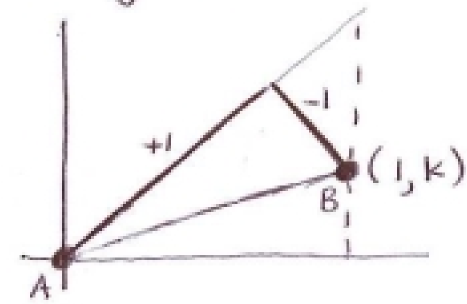
In general: $F = (y'^2 - 1)^2$, $F_{y'} = 4y'(y'^2 - 1)$

$$F_{y'y'} = 12(y'^2 - 1/3)$$

Euler equation: $\frac{d}{dx} F_{y'} = 0 \Rightarrow F_{y'y'} y'' = 0 \Rightarrow \begin{cases} F_{y'y'} = 0 \\ y'' = 0 \end{cases}$

$\therefore$ Extremals are straight lines, $y = ax + b$

<u>Case (a)</u> $0 \leq \phi < \frac{\pi}{4}$



Clearly, straight line from A to B is not the minimizing curve because $I \neq 0$ on the line $(|y'| \neq 1)$.

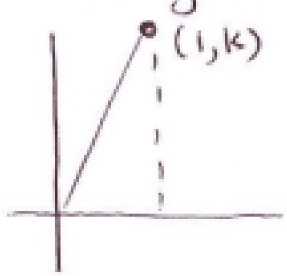
However $F_{y'y'} = 12(y'^2 - \frac{1}{3}) \geq 0$ on all straight lines with slope

$$\phi \ni \frac{\pi}{6} \leq \phi \leq \frac{\pi}{4} \quad (1/\sqrt{3} \leq k \leq 1)$$

$\rightarrow$ All the necessary conditions hold but no extremum.

<u>Case (b)</u> $\phi = \pi/4$: trivial ($y = x$ is minimizing curve).

<u>Case (c)</u> $\frac{\pi}{4} < \phi \leq \frac{\pi}{2}$ (i.e. $k > 1$) Now straight line is minimizing curve ($y = kx$); can reject lines of slope ± 1 on grounds of single-valuedness.



Consider $y = kx + h(x)$, $h(0) = h(1) = 0$.

Then $I(kx+h) - I(kx)$

$$= \int_0^1 (2kh' + h'^2)^2 dx + 2(k^2-1)\int_0^1 h'^2 dx > 0$$

$$\left(\int_0^1 h' dx = 0\right)$$

<u>Notation</u>: $I(y) = \int_a^b F(x,y,y')dx$; $y(a) = A, y(b) = B$

Suppose y is the minimizing function. Construct

$$Y(x) = y(x) + \overbrace{\varepsilon\eta(x)}^{\delta y}, \quad \eta(a) = \eta(b) = 0, \quad Y \in C'[a,b]$$

Set $\Psi(\varepsilon) = I(y+\varepsilon\eta)$, $\Rightarrow \Psi'(0) = 0$

$$Y' = y' + \underbrace{\varepsilon\eta'}_{\delta y'}, \quad \delta y' = \frac{d}{dx}\delta y \; ; \; \varepsilon \text{ small} \Rightarrow \left\{ \begin{matrix} \varepsilon\eta \\ \varepsilon\eta' \end{matrix} \right. \text{ small}$$

$$\Psi(\varepsilon) = I(y+\varepsilon\eta) = \int_a^b F(x, y+\varepsilon\eta, y'+\varepsilon\eta')\,dx$$

$$\Psi'(\varepsilon) = \lim_{\varepsilon' \to \varepsilon} \frac{I(y+\varepsilon'\eta) - I(y+\varepsilon\eta)}{\varepsilon' - \varepsilon} = \int_a^b (F_y\eta + F_{y'}\eta')dx$$

$\hookrightarrow$ evaluated at $y+\varepsilon\eta, y'+\varepsilon\eta'$

$$\Psi'(0) = \int_a^b (F_y\eta + F_{y'}\eta')dx \quad \text{(evaluated at } (x,y,y'))$$

$$\Psi'(0) = 0 \Rightarrow \frac{d}{dx}F_{y'} = F_y$$

<u>Defn</u>: $C_p^1[a,b]$ is the space of piecewise smooth, continuous functions on $[a,b]$

6. BROKEN EXTREMALS: WEIERSTRASS-ERDMAN CONDITIONS

(1) Defn Broken extremals are extremals which have a corner (i.e. continuous, with discontinuous slope).

⟨6.1⟩ 1st corner condition: Suppose y is extremal with discontinuity in y' at some point.

$$\Psi'(0) = \int_a^b (F_y \eta + F_{y'} \eta')\,dx$$
$$= \underbrace{\cancel{\eta} \int_a^x F_y \,d\xi}_{=0} + \int_a^b \left\{ F_{y'} - \int_a^x F_y \,d\xi \right\} \eta'(x)\,dx = 0$$

$$\xRightarrow{\text{Lemma (5.2)}} \quad F_{y'} - \int_a^x F_y \,d\xi = \text{constant} \quad \left(\begin{array}{c}\text{equ. of}\\ \text{DuBois-Raymond}\end{array}\right)$$

$$\therefore \quad F_{y'} = \int_a^x F_y \,d\xi + C \quad \text{, i.e. } F_{y'} \text{ is continuous}$$

i.e., even at a point where y' is discontinuous, x_0 say:

(1) $$\boxed{F_{y'}(x_0, y, y'_+) = F_{y'}(x_0, y, y'_-)}$$ 1st corner condition

⟨6.2⟩ 2nd corner condition: suppose minimizing curve $y = y(x)$ is given parametrically by $(x = t, y = y(t))$. Imbed the curve in the comparison family

$$x = t + \varepsilon\alpha(t), \quad y = y(t) \qquad (\alpha(a) = \alpha(b) = 0)$$

Consider
$$\Phi(\varepsilon) = I(x = t + \varepsilon\alpha(t), y = y(t)) -$$
$$= \int_a^b F\left(t + \varepsilon\alpha(t), y(t), \frac{\dot{y}(t)}{1 + \varepsilon\dot{\alpha}(t)}\right)(1 + \varepsilon\dot{\alpha}(t))\,dt$$

$$\Psi'(0) = \int_a^b \left(F_x \alpha - F_{y'} \dot{\alpha} \dot{y} + F \dot{\alpha} \right) dt$$

$$= \int_a^b \left[F_x \alpha + (F - \dot{y} F_{y'}) \dot{\alpha} \right] dt = 0$$

(note: $\dot{y} = y'$ because at $\varepsilon = 0$, we have $x = t$)

Working as before, it follows (cf. lemma 5.4):

$$\Psi'(0) = \int_a^b \left\{ F_x \alpha(x) + (F - y' F_{y'}) \alpha'(x) \right\} dx$$

$$= \alpha(x) \int_a^x F_x \, d\xi + \int_a^b \left[-\int_a^x F_x \, d\xi + (F - y' F_{y'}) \right] \alpha' dx$$

$$\Rightarrow \quad F - y' F_{y'} = \text{Constant} + \int_a^x F_x \, d\xi \quad \Rightarrow \text{continuous}$$

At a point x_0 where y' has a corner:

$$F(x_0, y, y'_+) - y'_+ F_{y'}(x_0, y, y'_+) =$$

$$= F(x_0, y, y'_-) - y'_- F_{y'}(x_0, y_0, y'_-) \quad \Rightarrow$$

(using 1st corner condition):

$$(2) \qquad \underbrace{F(x_0, y, y'_+) - F(x_0, y, y'_-) - (y'_+ - y'_-) \overset{\text{continuous}}{F_{y'}(x_0, y, y'_\pm)} = 0}_{\text{2nd corner condition}}$$

Ex.(1) (Ex. 5.4 revisited)

$$I(y) = \int_0^1 (y'^2-1)^2\,dx, \quad y(0)=0, \quad y(1)=k$$

$$F_{y'} = 4y'(y'^2-1)$$

$$F - y'F_{y'} = -(3y'^2+1)(y'^2-1)$$

Let P, Q be values of y' at each side of jump,

$$P = y'_+, \quad Q = y'_-$$

$$\left.\begin{array}{ll} (1^{st}\text{ cond}) & P(P^2-1) - Q(Q^2-1) = 0 \\ (2^{nd}\text{ cond}) & (3P^2+1)(P^2-1) - (3Q^2+1)(Q^2-1) = 0 \end{array}\right\}$$

$\therefore$ Solution: $(P,Q) = \{(+1,-1), (-1,+1)\}$

Get same result as before (gotten by inspection)

Ex.(2) Minimize $I(y) = \int_0^1 \left(y'^2 - \lambda\sqrt{1+y'^2}\right)dx$

$$y(0) = 0 = y(1)$$

Euler equ.: $\frac{d}{dx}F_{y'} = 0 \Rightarrow$ solution is $y = ax+b$

$\therefore$ all extremals are straight lines.

$\therefore y(x) \equiv 0$ is the only extremal in $C^1[0,1]$ that satisfies BC,

$$I(0) = -\lambda$$

Broken extremals (let $y' = \tan\vartheta$)

(1) Continuity of $F_{y'} = 2y' - \dfrac{\lambda y'}{\sqrt{1+y'^2}} = 2\tan\vartheta - \lambda\sin\vartheta$

(2) Continuity of $F - y'F_{y'} = -y'^2 - \dfrac{\lambda}{\sqrt{1+y'^2}} = -\tan^2\vartheta - \lambda\cos\vartheta$

If α and β denote values of ϑ to the left and right of a corner, the conditions (1-2) give

$$\left.\begin{cases} 2\tan\alpha - \lambda\sin\alpha = 2\tan\beta - \lambda\sin\beta \\ -\tan^2\alpha - \lambda\cos\alpha = -\tan^2\beta - \lambda\cos\beta \end{cases}\right\} \rightarrow \alpha + \beta = 0$$

Note: Combine (1) & (2) to get

(*) $\tan\alpha + \tan\beta = -2\dfrac{\cos\alpha - \cos\beta}{\sin\alpha - \sin\beta}$

then use half-tangent formulas ($t = \tan\vartheta/2$) $\longrightarrow \begin{cases} \sin\vartheta = \dfrac{2t}{1+t^2} \\ \cos\vartheta = \dfrac{1-t^2}{1+t^2} \end{cases}$ $\tan\vartheta = \dfrac{2t}{1-t^2}$

If $\tan\alpha/2 = a$, $\tan\beta/2 = b$, can show (with some effort) that ~~the~~ (*) gives $\Rightarrow (a+b)(a-b)^3 = 0$. But $a = b$ implies $\tan\alpha/2 = \tan\beta/2$ i.e. smooth solution; for jump, take $a + b = 0 \Rightarrow \tan\frac{\alpha}{2} = -\tan\frac{\beta}{2} \Rightarrow \alpha + \beta = 0$

Then, first equation gives $2\tan\alpha = \lambda \sin\alpha \Rightarrow$

$$\cos\alpha = 2/\lambda$$

Since $|\cos\alpha| \le 1 \Rightarrow$ we have a solution of (1-2)′ with $\alpha \neq \phi$ if $2/\lambda \le 1$.

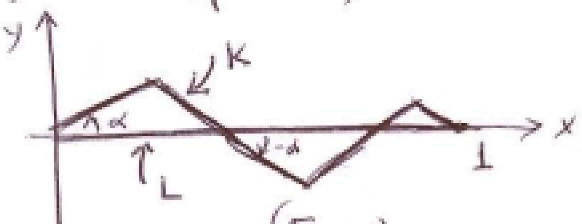
$\therefore$ If $\lambda \ge 2$ we can have a corner on a minimizing solution (with slopes $\tan\alpha$ & $-\tan\alpha$) where $\cos\alpha = 2/\lambda$, $\tan\alpha = \sqrt{\frac{\lambda^2}{4} - 1}$

while if $0 < \lambda < 2$, no corner is possible.

$K: y' = \tan\alpha$

$\Downarrow$

$y'^2 = \frac{\lambda^2}{4} - 1$



(Fig. 1)

$$\begin{cases} I(L) = -\lambda \\ I(K) = -1 - \lambda^2/4 \end{cases}$$

Consider a broken line extremal and also the $C'[0,1]$ extremal $y \equiv 0$.

$$\therefore I(L) - I(K) = \left(\tfrac{1}{2}\lambda - 1\right)^2 > 0$$

$\therefore$ when broken-line extremal exists (i.e. for $\lambda \ge 2$) we always have

$$I(K) < I(L)$$

Furthermore, $F_{y'y'} = 2 - \frac{\lambda}{(1+y'^2)^{3/2}} = 2 - \lambda\cos^3\theta \ge 0$ if $\lambda \le 2$

~~i.e. straight line extrema are minim~~

But if $\lambda > 2$ we have $F_{y'y'} < 0$ on L

i.e. L cannot provide even a weak relative minimum in this case.

But on k, $F_{y'y'} = \frac{2}{\lambda^2}(\lambda^2-4) \geq 0$ $\left(\lambda>2 \text{ for } k \text{ to exist}\right)$

$\therefore$ Legendre condition satisfied on k.

Good guess: $\begin{cases} L \text{ is minimizing curve if } \lambda \leq 2 \\ k \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{if } \lambda > 2 \end{cases}$

Let Γ be any curve $\in C_p^1[0,1]$ joining $(0,0)$ to $(1,0)$. Denote $y'(x)$ on this curve by $\tan\phi(x)$. Then, if $\lambda<2$,

$$I(\Gamma) - I(L) = \int_0^1 (\tan^2\phi - \lambda\sec\phi)\,dx - \int_0^1 (-\lambda)\,dx$$

$$= \int_0^1 (\underbrace{\tan^2\phi}_{\sec^2\phi - 1} - \lambda\sec\phi + \lambda)\,dx = \int_0^1 (\sec\phi - 1)(\sec\phi + 1 - \lambda)\,dx > 0$$

$\left(\text{since } \sec\phi \geq 1,\ \lambda < 2\right)$ $\therefore I(L) - I(\Gamma) \leq 0$ if $\lambda<0$

If $\lambda>2$ consider $I(k)-I(L)$ where k is the broken line extremal in (Fig. 1). Then

$\cancel{I(k) - I(L) = \int_0^1 [(\sec^2\phi - \tfrac{1}{2}\lambda)^2}$

$$I(\Gamma) - I(k) = \int_0^1 \left[\sec^2\phi - \lambda\sec\phi - \left(-1 - \tfrac{1}{4}\lambda^2\right)\right]dx$$

$$= \int_0^1 (\sec\phi - \lambda/2)^2\,dx \geq 0$$

PROBLEM SET I

1. Disuss the upper and lower semicontinuity of the arclength functional $\int_0^b \sqrt{1+y'^2}\,dx$ with respect to strong variations in the space $C'[a,b]$

2. Discuss the problem of minimizing $\int_0^{\alpha} (y'^4 - 6y'^2)\,dx$, $y(0)=0$, $y(\alpha)=\beta$. Consider both $C'[0,\alpha]$ and $C'_p[0,\alpha]$ and comment (with reasons) on whether your answers are weak or strong minima.

3. Determine whether it is possible to show that every equation $y''=f(x,y,y')$ is the Euler equation for some functional $\int F(x,y,y')\,dx$. How do we determine the function $F(x,y,y')$ from the function $f(x,y,y')$? Determine whether it is possible to find all functionals for which the extremals are straight lines.

4. Consider (i) $\int_{x_0}^{x_1} (ay'^2 + byy' + cy^2)\,dx$, $y(x_0)=y_0$, $y(x_1)=y_1$, $a \neq 0$.
   (ii) $\int_{x_0}^{x_1} y'^3\,dx$, $y(x_0)=y_0$, $y(x_1)=y_1$

   Can these functionals have broken extremals? If so, find them!

5. Find the function for which the following functional may have a weak extremum:

$$\int_0^1 (y''^2 - 2xy)\,dx, \quad y(0)=y'(0)=0\,;\ y(1)=\frac{1}{120}$$

($y'(1)$ is not prescribed).