where $\mathbf{p} = \operatorname{grad} \varphi(x, y)$ in a region $G$, or

2) $$Q(\mathbf{p}, \mathbf{p}) = \iint_G \mathbf{p}^2 \, dx \, dy$$

where $\mathbf{p} = \Delta\varphi(x, y)$, or a form such as

3) $$Q(\mathbf{p}, \mathbf{p}) = \int_0^1 \int_0^1 K(x, y)p(x)p(y) \, dx \, dy .$$

The definition of the corresponding polar forms $Q(\mathbf{p}, \mathbf{q})$ is obvious.

We interpret $Q(\mathbf{p}, \mathbf{p})$ as the square of the length of $\mathbf{p}$, and call vectors $\mathbf{p}$, $\mathbf{q}$ orthogonal if

$$Q(\mathbf{p}, \mathbf{q}) = 0.$$

For orthogonal vectors $\mathbf{p}$, $\mathbf{q}$ we have

$$Q(\mathbf{p} + \mathbf{q}, \mathbf{p} + \mathbf{q}) = Q(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q}) = Q(\mathbf{p}, \mathbf{p}) + Q(\mathbf{q}, \mathbf{q})$$

("Pythagorean Theorem").

If $\Omega$ is a linear subspace of the vector space $\Lambda$, we can define another linear subspace $\Sigma$ orthogonal to $\Omega$ as the space of all vectors $\boldsymbol{\sigma}$ orthogonal to all vectors $\boldsymbol{\omega}$ of $\Omega$:

$$Q(\boldsymbol{\omega}, \boldsymbol{\sigma}) = 0.$$

In all instances considered here the following facts are true: if $\Sigma$ is the linear subspace orthogonal to $\Omega$, then $\Omega$ is the linear subspace orthogonal to $\Sigma$. Every element $\mathbf{p}$ in $\Lambda$ can be uniquely decomposed into the sum of "projections" $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$:

$$\mathbf{p} = \boldsymbol{\omega} + \boldsymbol{\sigma} \qquad (\boldsymbol{\omega} \text{ in } \Omega,\ \boldsymbol{\sigma} \text{ in } \Sigma).$$

If we restrict our vectors by imposing suitable continuity and differentiability conditions, then the theorems which guarantee the existence of a solution for our variational problems also insure the validity of our statements. Accordingly, although we then operate in an incomplete Hilbert space, we assume that the projection theorem holds for the respective incomplete spaces.

We now consider two variational problems.

I: Given two vectors $\mathbf{p}_0$ and $\mathbf{q}_0$ in $\Lambda$, minimize

$$Q(\mathbf{p}, \mathbf{p}) - 2Q(\mathbf{p}, \mathbf{q}_0)$$

admitting for competition vectors $\mathbf{p}$ for which $\mathbf{p} - \mathbf{p}_0$ is in a prescribed linear subspace $\Omega$. Note that the difference of two admissible