Hamilton's principle, employing the expression (99) for the kinetic energy. We obtain the equations

$$\mu\Delta\Delta u + \rho u_{tt} = 0$$

or, more generally,

$$\mu\Delta\Delta u + \rho u_{tt} + f(x, y, t) = 0$$

with appropriate boundary conditions just as above in the equilibrium problems. For the characterization of an actual physical problem these boundary conditions have to be supplemented by initial conditions which describe the initial state, i.e. the functions $u(x, y, 0)$ and $u_t(x, y, 0)$.

§11. *Reciprocal Quadratic Variational Problems*[1]

Linear functional equations of mathematical physics result from quadratic variational problems. It is illuminating, especially in regard to the reciprocity phenomenon discussed in §9, to consider such quadratic problems from a somewhat general and abstract viewpoint (compare Ch. I, §5, 3) and at the same time to use suggestive geometrical language.[2]

We consider, in a linear vector space Λ of vectors $\mathbf{p}, \mathbf{q}, \cdots$, a quadratic form $Q(\mathbf{p}, \mathbf{p})$ which we may assume to be positive definite. The vectors may be functions or systems of functions; for example

$$Q(\mathbf{p}, \mathbf{p}) = \iint_G \mathbf{p}^2 \, dx \, dy \tag{1}$$

[1] This section has been added for the present edition.

[2] Recently several authors have rediscovered and advanced the theories indicated in §9. See J. L. Synge, The method of the hypercircle in function-space for boundary value problems, Proc. Roy. Soc. London, Ser. A, Vol. 191, 1941, pp. 447–467; W. Prager and J. L. Synge, Approximations in elasticity based on the concept of function space, Q. App. Math., Vol. 5, 1947, pp. 241–269; J. L. Synge, The method of the hypercircle in elasticity when body forces are present, Q. App. Math., Vol. 6, 1948, pp. 15–19 and the literature quoted there. Synge's papers have called attention to the advantage of a geometrical interpretation from which the distance of the approximations to the exact solutions can be deduced. See also J. B. Diaz and H. J. Greenberg, Upper and lower bounds for the solution of the first boundary value problem of elasticity, Q. App. Math., Vol. 6, 1948, pp. 326–331; H. J. Greenberg, The determination of upper and lower bounds for the solution of the Dirichlet problem, J. Math. Phys., Vol. 27, 1948, pp. 161–182; J. B. Diaz and H. J. Greenberg, Upper and lower bounds for the solution of the first biharmonic boundary value problem, J. Math. Phys., Vol. 27, 1948, pp. 193–201; J. B. Diaz, Upper and lower bounds for quadratic functionals, Proc. of the Symposium on Spectral Theory and Differential Problems, Stillwater, 1951.