$\mathbf{p}$ and $\mathbf{q}$ are chosen, the sphere of radius $|(\mathbf{p} - \mathbf{q})/2|$ about the point $(\mathbf{p} + \mathbf{q})/2$ is a geometric locus for $\mathbf{u}$.

Special cases (see also §12, 12) are easily fitted into the general scheme. For example, as we have seen, Dirichlet's problem for the harmonic differential equation in a region G with boundary Γ (see §8) corresponds to $Q(\mathbf{p}, \mathbf{p}) = \iint_G \mathbf{p}^2\, dx\, dy$. The space Ω_0 of $\mathbf{p}$ is defined by $\mathbf{p} = \operatorname{grad} \varphi(x, y)$, $\varphi - \varphi_0 = 0$ on Γ, $\mathbf{p}_0 = \operatorname{grad} \varphi_0(x, y)$, where φ_0 is a prescribed function. The space $\Sigma_0 = \Sigma$ of $\mathbf{q}$ is defined by $\operatorname{div} \mathbf{q} = 0$, $\mathbf{q}_0 = 0$. Incidentally, it should be noted that in problem II the expression $Q(\mathbf{q}, \mathbf{p}_0)$ can be transformed into the boundary integral $\int_\Gamma \varphi_0 q_n\, ds$ where q_n is the normal component of $\mathbf{q}$.

Thus the reciprocal problem II can be formulated if we merely know the prescribed boundary values of φ; no explicit knowledge of a function $\varphi_0(x, y)$ is necessary.

A similar fact is true of other examples, e.g. the problem of the clamped plate, where

$$Q(\mathbf{p}, \mathbf{p}) = \iint_G \mathbf{p}^2\, dx\, dy,$$

$$\mathbf{p} = \Delta\varphi, \qquad \mathbf{p}_0 = \Delta\varphi_0, \qquad \Delta\mathbf{q} = 0, \qquad \mathbf{q}_0 = 0,$$

and where $\varphi - \varphi_0$ and its normal derivative are supposed to vanish on Γ. Since

$$Q(\mathbf{q}, \mathbf{p}_0) = \int_\Gamma \left(\varphi_0 \frac{\partial \mathbf{q}}{\partial n} - \frac{\partial \varphi_0}{\partial n} \mathbf{q} \right) ds,$$

where $\partial/\partial n$ denotes the normal derivative, problem II actually refers only to the given boundary values of φ and $\partial\varphi/\partial n$.

§12. *Supplementary Remarks and Exercises*

1. Variational Problem for a Given Differential Equation. For a given ordinary second order differential equation $y'' = f(x, y, y')$ one can always find a function $F(x, y, y')$ such that the equation $[F]_y = 0$, when solved for y'', is identical with the differential equation.[1]

[1] Cf. O. Bolza, Vorlesungen über Variationsrechnung, Teubner, Leipzig and Berlin, 1909, pp. 37–39.