On a Lagrange problem

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Abstract The authors establish existence and give the explicit formulas for the solution of the classical Lagrange problem of determining the shape of column which has the largest critical buckling load with the Dirichlet boundary conditions and some its generalizations. Our results justify the Keller-Tadjbakhsh result in [3]. The developped methods allow to study the problem with other boundary conditions and are of interest for other variational problems.

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1. Introduction

The considered Lagrange problem consists in the finding of the extremal values of the functional

$$L[Q,y] = \frac{\int_0^1 Q(x)y''(x)^2dx}{\int_0^1 y'(x)^2dx}$$

under the conditions: $y \in H^2(0,1)$, $Q(x)$ is a bounded measurable function,

$$\int_0^1 Q(x)^\alpha dx = 1, \; Q(x) \geq 0, \tag{1}$$

$$y(0) = 0, \; y'(0) = 0, \; y(1) = 0, \; y'(1) = 0, \tag{2}$$

where $\alpha \in R \setminus \{0\}$. We are looking for the value

$$M_\alpha = \sup_Q \inf_y L[Q,y]$$

and for the optimal functions Q and y. It is easy to see that this problem is equivalent to the variational problem on the extremum of the functional

$$F[Q,y] = \frac{\int_0^1 Q(x)y'(x)^2dx}{\int_0^1 y(x)^2dx},$$

under the conditions (1) and

$$y(0) = 0, \; y(1) = 0, \; \int_0^1 y(x)dx = 0.$$

The Euler-Lagrange equation for the functional L has the form

$$(Q(x)y'')'' + \lambda y'' = 0, \; y(0) = 0, \; y'(0) = 0, \; y(1) = 0, \; y'(1) = 0.$$

This problem is very important for applications. For example, it is essential in finding the strongest column of a given volume (the most important values are then $\alpha = 1/2$ or $1/3$) and was considered by many authors (see, for example, [1]-[9]). The authors of the articles [4]-[9] used methods of the functional analysis and of the variational calculus, sometimes very complicated. However, the problem has not been solved until now.

Let us reproduce the arguments of the Keller-Tadjbakhsh article [3]. Let us suppose that there exists a function $Q_0(x)$ that maximizes the lowest eigen-value. Let $Q(x,\varepsilon)$ be a family of functions that depend smoothly on ε and such that $Q(x,0) = Q_0(x)$. Assume that λ and y, the lowest eigen-value and the corresponding eigenfunction with $Q = Q(x,\varepsilon)$, also depend smoothly on ε. Then we may differentiate the equation

$$(Q(x,\varepsilon)y''(x,\varepsilon))'' + \lambda(\varepsilon)y''(x,\varepsilon) = 0$$

with respect to ε and obtain the equation

$$(Q_0(x)z'')'' + \lambda z'' + (Q_1(x)y'')'' + \mu y'' = 0,$$

where $Q_1(x) = \partial Q(x,0)/\partial\varepsilon$, $z(x) = \partial y(x,0)/\partial\varepsilon$, and $\mu = \partial\lambda(0)/\partial\varepsilon$. We multiply the first equation by z, the second by y, subtract one from the other, and integrate the result over $[0,1]$. In virtue of the boundary conditions we have

$$\int_0^1((Q_0(x)y'')''z - (Q_0(x)z'')''y)dx = 0,$$

$$\int_0^1(y''z - z''y)dx = 0.$$

Therefore

$$\int_0^1 Q_1(x)y''(x)^2dx - \int_0^1 \mu y'(x)^2dx = 0.$$

Since λ is the maximal value of $\lambda(\varepsilon)$, we see that $\mu = 0$ and so

$$\int_0^1 Q_1(x)y''(x)^2dx = 0$$

and this is true for any function $Q_1(x)$ such that

$$\int_0^1 Q_0(x)^{\alpha-1}Q_1(x)dx = 0.$$

Thus we have as a necessary condition for a maximum the relation $y''(x)^2 = CQ_0(x)^{\alpha-1}$. This leads to a nonlinear equation for y

$$(|y''|^{2/(\alpha-1)}y'')'' + \lambda y'' = 0,$$

which is integrable. Indeed, if we set $y'' = z$, then

$$(|z|^{2/(\alpha-1)}z)'' + \lambda z = 0,$$

and if we set now $z' = P(z)$ we obtain a linear equation of first order for P^2. The weak point of this proof is that the function $\lambda(\varepsilon)$ can be nonregular, because the lowest eigen-value λ can be double. Besides, the existence of the optimal solution was never proved.

The authors of [9] claimed to prove that the result of Keller and Tadjbakhsh in [3] is not correct. However, their calculations are erroneous and the value $16\pi^2/3$ found in [3] is optimal. [3]

We propose here another approach, which allows us to prove that the indicated solution is really optimal and gives a globally extremal value to the functional L. Let us remark that we use the Sobolev's type spaces $W_p^l(0,1)$ with $l = 1, 2$ and any real values of $p \neq 0$, what is also interesting outside of the frame of the Lagrange problem. Furthermore, we prove the existence of the optimal solution. The obtained results can be extended to the multi-dimensional case also. Close results for functionals depending on y, y' only have been obtained in our works [10]-[12].

The important role in what follows belongs to the functional

$$G[y] = \frac{(\int_0^1 |y'(x)|^p dx)^{2/p}}{\int_0^1 y(x)^2 dx}, \ where \ p = \frac{2\alpha}{\alpha - 1}.$$

Let R_α be the set of bounded measurable functions Q defined on $[0,1]$ satisfying the conditions (1).

Let $K_p(a,b)$ for real $p \neq 0$ be the set of nondecreasing real functions y defined on $[a,b]$, absolutely continuous on $[a, b-\varepsilon]$ for any $\varepsilon > 0$, and such that $y(a) \geq 0$,

$$\int_a^b y'(x)^p dx < \infty, \ \int_a^b y(x)^2 dx < \infty.$$

Let $K_p(a,b,c)$ be the set of real functions y defined on $[a,c]$ and such that $y \in K_p(a,b)$, $y(-x) \in K_p(-c,-b)$, $\int_a^b |y'(x)|^p dx < \infty$ and $\int_b^c |y'(x)|^p dx < \infty$.

[3]The functions $u_{2,0}$ and $\tilde{u}$ on the page 321 in [9] satisfy the same equation for $0 < x < 1/4$, so that $L[A, u_{2,0}] = L[A, \tilde{u}] = 16\pi^2/3$ in conradiction to the authors results. The proof of Theorem 2.2 on the page 291 is based on the assumption that $\overline{v} \neq 0$, which is not true, if the eigenfunctions corresponding to λ_1 are odd.

Let $\mathcal{H}$ be the set of functions y belonging to $H^2(0,1)$ and satisfying conditions (2).

Finally set

$$m_\alpha = \inf_{Q\in R_\alpha} \inf_{y\in\mathcal{H}} L[Q,y],\ M_\alpha = \sup_{Q\in R_\alpha} \inf_{y\in\mathcal{H}} L[Q,y].$$

Our aim is to find the values of m_α, M_α and the functions Q, y realizing these extremal values.

2. Main results

Our main result is the following.

Theorem . *If* $0 < \alpha < 1$, *then*

$$M_\alpha = 4\frac{(2\alpha+1)}{\alpha}\Big(\frac{1+\alpha}{1+2\alpha}\Big)^{1-1/\alpha} B(\frac{1}{2},\frac{1}{2}+\frac{1}{2\alpha})^2$$

and $m_\alpha = 0$.

There exist two functions $Q_0 \in R_\alpha$ *and* $y_0 \in \mathcal{H}$ *such that*

$$\inf_y L[Q_0,y] = L[Q_0,y_0] = M_\alpha.$$

Moreover, $Q_0(x) = Q_0(1-x)$ *for* $0 \le x \le 1$, $Q_0(x) = Q_0(1/2-x)$ *for* $0 \le x \le 1/2$, $Q_0(x) = |y_0''(x)|^{2/(\alpha-1)}$, *and the function* $z(x) = y_0'(x)$ *is such that*

$$\frac{(\int_0^{1/4}|z'(x)|^p dx)^{2/p}}{\int_0^{1/4} z(x)^2 dx} = \sup_{y\in K_p(0,1/4)} \frac{(\int_0^{1/4}|y'(x)|^p dx)^{2/p}}{\int_0^{1/4} y(x)^2 dx},$$

where $p = 2\alpha/(\alpha-1)$.

The proof is given ih the following Section.

Remark. In particular, in the classical Lagrange problem with $\alpha = 1/2$ the optimal function $Q_0(x)$ is defined on $(0,1)$ in the parametric form as follows

$$x = (2t + sin\, 2t)/4\pi,\ Q_0(x) = 16\ cos^4\, t/9;\ 0 \le t \le 2\pi.$$

This function and the optimal value $M = 16\pi^2/3$ have been indicated by Keller-Tadjbakhsh in [3]. The optimal column has two points at which $Q(x)$ vanishes.

3. Proofs

Lemma 1. *Let $p < 0$. Then the function*

$$F(x,y) = x^{2/p}y^{3-2/p} + (1-x)^{2/p}(1-y)^{3-2/p},\ 0 \le x \le 1,\ 0 \le y \le 1$$

has a minimal value $F_{\min} = 1/4$ at $x = y = 1/2$.

Proof. Observe that $2/p < 0$ and $3 - 2/p > 0$. Then for $y \neq 0, 1$

$$\lim_{x\to+0} F(x,y) = +\infty,\ \lim_{x\to 1-0} F(x,y) = +\infty.$$

If $y = 0$, then $F(x,0) = (1-x)^{2/p} \ge 1$ and for $y = 1$ we have $F(x,1) = x^{2/p} \ge 1$. Therefore, the values of F on the boundary of the square are ≥ 1.

If $x < \delta, y < \delta$, then

$$F(x,y) \ge (1-x)^{2/p}(1-y)^{3-2/p} \ge (1-y)^{3-2/p} \ge (1-\delta)^{3-2/p}$$

and therefore $F(x,y) > 3/4$, if δ is small enough. The same is true for a small neighbourhood of the points $(0,1)$, $(1,0)$ and $(1,1)$.

On the other hand, we have $F(1/2, 1/2) = 1/4$.

Therefore, the function F has an inner minimum point (x_0, y_0). We have at this point

$$\partial F(x_0,y_0)/\partial x = 2/p[x_0^{2/p-1}y_0^{3-2/p} - (1-x_0)^{2/p-1}(1-y_0)^{3-2/p}] = 0;$$

$$\partial F(x_0,y_0)/\partial y = (3-2/p)[x_0^{2/p}y_0^{2-2/p} - (1-x_0)^{2/p}(1-y_0)^{2-2/p}] = 0.$$

Then

$$\begin{aligned} x_0^{2/p-1}y_0^{3-2/p} &= (1-x_0)^{2/p-1}(1-y_0)^{3-2/p}, \\ x_0^{2/p}y_0^{2-2/p} &= (1-x_0)^{2/p}(1-y_0)^{2-2/p}. \end{aligned} \tag{3}$$

Dividing these equalities term by term we obtain that

$$y_0/x_0 = (1-x_0)/(1-y_0),$$

i.e. $x_0 = y_0$. Then (3) implies that $x_0^2 = (1-x_0)^2$ and thus $x_0 = 1/2$. □

Lemma 2. *Let p be a negative number. Then for all functions $y \in K_p(0,h)$ the following estimate is valid:*

$$\left(\int_0^h y'(x)^p dx\right)^{1/p} \le \left(\frac{h}{4}\right)^{1/p-3/2} \left(\int_0^h y(x)^2 dx\right)^{1/2}.$$

Proof. Let at first $h = 1$, and $\int_0^1 y'(x)^p dx = 1$. Then $\int_0^{1/2} y'(x)^p dx < 1$. Let E be the subset of points x in $[0, 1/2]$ such that $y'(x) > 4^{1/p}$ and μ its measure. Then $y'(x)^p > 4$ on a set of measure $1/2 - \mu$ IN $[0, 1/2]$ and

$$1 > 4(1/2 - \mu),$$

i.e. $\mu > 1/4$. Therefore

$$y(1/2) \geq \int_0^{1/2} y'(x)dx > \int_E y'(x)dx > 4^{1/p-1}.$$

Since y is increasing, we have $y(x) > 4^{1/p-1}$ for $x > 1/2$. Then

$$\int_0^1 y(x)dx > \int_{1/2}^1 y(x)dx > 4^{1/p-3/2}.$$

Therefore

$$1 = (\int_0^1 y'(x)^p dx)^{1/p} \leq 4^{3/2-1/p} \int_0^1 y(x)dx \leq 4^{3/2-1/p} (\int_0^1 y^2(x)dx)^{1/2}.$$

Let $h \neq 1$ and $y \in K_p(0, h)$. Then the function $z(x) = y(hx) \in K_p(0, h)$ and we can apply the proved estimate to the function z, so that

$$(\int_0^1 z'(x)^p dx)^{1/p} \leq 4^{3/2-1/p} (\int_0^1 z^2(x)dx)^{1/2}.$$

Thus

$$(\int_0^h y'(x)^p dx)^{1/p} \leq (4/h)^{3/2-1/p} (\int_0^h y^2(x)dx)^{1/2}$$

and the proof is complete. $\square$

Lemma 3. *Let $p < 0$, and $m = \sup_{y \in K_p(0,h)} G[y]$. Then*

$$m = \frac{1}{4}\Big(\frac{2-2p}{2-3p}\Big)^{2/p}(3 - \frac{2}{p})h^{2/p-3}B(1/2, 1-1/p)^2$$

and there exists a function $u_0 \in K_p(0, h)$ such that $G[u_0] = m$. Moreover, as $x \to h^-$ we have

$$u_0(x) = u_0(h) + c_1(h-x)^{p/(p-1)}[1 + o(1)], \; u_0'(x) = c_2(h-x)^{1/(p-1)}[1 + o(1)].$$

Proof. Let at first $h = 1$ and $\{y_k\}$ be a sequence of functions in $K_p(0,1)$ such that $\int_0^1 y_k'(x)^p dx = 1$ and $G[y_k] \to m$. In virtue of Lemma 2 the value of m is finite and positive.

Let us show that the functions y_k can be taken from $C_0^\infty(0,1)$.

Let y be a function of $K_p(0,1)$ and $y(0) = 0$. Set $y_1(x) = 0$ for $x < 0$, $y_1(x) = y(x)$ for $0 < x < 1-\varepsilon$ and $y_1(x) = y(1-\varepsilon) + \varepsilon^{-1/2p}(x - 1 + \varepsilon)$ for $x > 1-\varepsilon$. Let $u_k(x) = y_1(x) + \varepsilon_k x$, where $\varepsilon_k \to 0+$ as $k \to \infty$. Obviously, $\int_0^1 |u_k(x) - y(x)|^2 dx \to 0$, $u_k'(x)^p \leq y_1'(x)^p$ and by the Lebesgue theorem, $\int_0^1 |u_k'(x)|^p dx \to \int_0^1 |y'(x)|^p dx$. So we can assume that $y_k'(x) > \varepsilon_k > 0$.

Let once again y be a function of $K_p(0,1)$, $y(0) = 0$, and $y'(x) > \varepsilon > 0$. Set $z_k(x) = y_k(x) - y_k(0)$, where y_k is the averaging of y with a positive kernel so that

$$y_k(x) = k \int K(k(x-t)) y(t) dt, \int K(t) dt = 1, \; K(t) \geq 0, \; K \in C_0^\infty(-1,1).$$

Observe that $|y_k(0)| \leq y(1/k) = \delta_k \to 0$ as $k \to \infty$ so that z_k converges to y uniformly and

$$\int_0^1 |z_k(x) - y(x)|^2 dx \to 0, \quad \int_0^1 |z_k'(x) - y'(x)| dx \to 0.$$

Therefore, a subsequence z_{n_k}' converges to y' almost everywhere and $z_{n_k}'^p$ converges to y'^p almost everywhere. Since $0 \leq y(x) \leq C$ for $0 \leq x \leq 1$ and $y'(x) > 0$, we see that $\int_0^1 |y'(x)| dx \leq C$. Since $|z_{n_k}'|^p \leq \varepsilon^p$, by the Lebesgue theorem $\int_0^1 z_{n_k}'(x)^p dx \to \int_0^1 y'(x)^p dx$. This allows us to assume that all y_k are C^∞ functions.

If y_k is convex in an interval (x_1, x_2), where $0 \leq x_1 \leq x_2 \leq 1$, then it is possible to replace it by the linear function

$$z(x) = y_k(x_1) + \gamma(x - x_1), \; where\; \gamma = [y_k(x_2) - y_k(x_1)]/(x_2 - x_1)$$

so that the value of the functional G will increase. Indeed, then $y_k(x_1) = z(x_1)$, $y_k(x_2) = z(x_2)$ and

$$\int_{x_1}^{x_2} y_k(x)^2 dx \geq \int_{x_1}^{x_2} z(x)^2 dx.$$

On the other hand, by the Hölder inequality

$$x_2 - x_1 = \int_{x_1}^{x_2} y_k'(x)^{p/(p-1)} y_k'(x)^{p/(1-p)} dx$$

$$\leq (\int_{x_1}^{x_2} y_k'(x)dx)^{p/(p-1)}(\int_{x_1}^{x_2} y_k'(x)^p dx)^{1/(1-p)}$$

and therefore,

$$\int_{x_1}^{x_2} y_k'(x)^p dx \geq (\int_{x_1}^{x_2} y_k'(x)dx)^p (x_2 - x_1)^{1-p}$$
$$= [y_k(x_2) - y_k(x_1)]^p (x_2 - x_1)^{1-p}.$$

Since

$$\int_{x_1}^{x_2} z'(x)^p dx = \gamma^p (x_2 - x_1) = [y_k(x_2) - y_k(x_1)]^p (x_2 - x_1)^{1-p},$$

we see that $G[z] \geq G[y_k]$, if z coincides with y_k outside the interval (x_1, x_2).

We can do the same for all other intervals where y_k is convex. As the result we shall obtain the function z_k that can be described in the following way. Consider the set of points (x, y) such that $0 < x < 1$ and $y > y_k(x)$, and take the convex hull of this set. The lower boundary of the hull serves as the graph of the function z_k. Let us show that $z_k \in C^1(0, 1)$. Indeed, if a point $(x, z_k(x))$ belongs to an interval of the straight line, then it is obvious that z_k is smooth at the point x. The same is true in the case when the point $(x, z_k(x))$ belongs to a part of the graph of the function y_k. If z_k is linear on one side of x and coincides with y_k on the other side, then z_k is regular at x since its graph is lying on one side of the straight line, obtaining by the continuation of the linear function. At last, if the point $(x, z_k(x))$ is a limit point for a sequence of such points, then it is also a limit point for a sequence of points belonging to the graph of y_k and the derivative $z_k'(x)$ exists.

Therefore, if one changes each function y_k by a concave function z_k in the indicated way, then the sequence of these new functions z_k will be maximizing, too. It allows us to consider the maximizing sequence as a sequence of monotone concave functions, i.e. to suppose that the functions $y_k(x)$ and their derivatives $y_k'(x)$ are increasing.

For large k we have

$$\int_0^1 y_k(x)^2 dx \leq m^{-1} + 1.$$

Let $\varepsilon > 0$ be small enough. There is a point $\theta_k \in (1-\varepsilon, 1)$ such that $|y_k(\theta_k)|^2 \leq (m^{-1}+1)\varepsilon^{-1}$. Since the functions y_k are monotone, they are uniformly bounded for $0 \leq x \leq 1 - \varepsilon$. It yields

$$\int_{1-2\varepsilon}^{1-\varepsilon} y_k'(x)dx \leq C$$

with a constant C independent of k. Therefore, there exists a $\tilde{\theta}_k$ such that $1-2\varepsilon < \tilde{\theta}_k < 1-\varepsilon$ and $y_k'(\theta_k) \le C/\varepsilon$. Therefore the sequence $\{y_k'(x)\}$ is uniformly bounded in $[0, 1-2\varepsilon]$. Since $\int_0^1 y_k'(x)^p dx = 1$ there exists a $\theta_k' \in (0,\varepsilon)$ such that $|y_k'(x)(\theta_k')|^p \le 1/\varepsilon$ and therefore $|y_k'(x)|^p \le 1/\varepsilon$ for $\varepsilon \le x \le 1-2\varepsilon$.

By the Arzela theorem one can choose a uniformly in $[\varepsilon, 1-2\varepsilon]$ converging subsequence $\{y_{n_k}(x)\}$, and by Helly theorem one can suppose that the sequence $\{y_{n_k}'(x)\}$ converges almost everywhere in $[\varepsilon, 1-2\varepsilon]$. Using diagonalization, one can find a subsequence converging in $(0,1)$ to a function $u_0 \in K_p(0,1)$ such that

$$y_{n_k} \to u_0,\ y_{n_k}'^p \to u_0'^p$$

everywhere in $(0,1)$. By the Fatou theorem we have

$$\int_0^1 u_0(x)^2 dx \le m^{-1},\ \int_0^1 u_0'(x)^p dx \le 1.$$

Therefore

$$\Big(\int_0^1 y_0'(x)^p dx\Big)^{2/p} \ge 1,$$

so that $G[u_0] \ge m$. However, m is the maximal possible value of G, so that $\int_0^1 u_0(x)^2 dx = m^{-1}$ and $\int_0^1 u_0'(x)^p dx = 1$.

The Euler-Lagrange equation has the form

$$(u_0'^{p-1})' + m u_0 = 0,\ u_0(0) = 0$$

so that

$$(p-1)u_0'^{p-1} u_0'' + m u_0 u_0' = 0$$

or

$$u_0'^p + m_1 u_0^2 = C,$$

where $m_1 = mp/[2(p-1)] > 0$. Integrating this equality over $(0,1)$, we obtain that

$$C = 1 + m_1 m^{-1} = 1 + p/2(p-1) > 0.$$

Therefore

$$\int_0^{u_0} \frac{dz}{(C - m_1 z^2)^{1/p}} = x.$$

Repeating the same arguments as in the proof of the preceding Lemma, we obtain that $u_0'(1) = \infty$. Let $M = \max u_0(x) = u_0(1)$. Then $M^2 m_1 = C$, i.e.

$$M = \Big(\frac{C}{m_1}\Big)^{1/2} = \Big(\frac{3p-2}{mp}\Big)^{1/2}$$

and

$$\int_0^{M} \frac{dz}{(C - m_1 z^2)^{1/p}} = 1.$$

Set $z = (C/m_1)^{1/2} t$ so that

$$\int_0^1 \frac{dt}{(1-t^2)^{1/p}} = C^{1/p-1/2} m_1^{1/2} = \frac{(2-3p)^{1/p-1/2}(-pm)^{1/2}}{(2-2p)^{1/p}},$$

and therefore

$$m^{1/2} = \frac{(2-2p)^{1/p}}{2(2-3p)^{1/p-1/2}(-p)^{1/2}} B(\frac{1}{2}, 1 - \frac{1}{p}).$$

Since

$$\int_{u_0}^{M} \frac{dz}{(C - m_1 z^2)^{1/p}} = 1 - x,$$

we can see that $u_0(x) = M + c_1(1-x)^{\gamma}[1 + o(1)]$ as $x \to 1$, where $\gamma = p/(p-1), c_1 < 0$.

In order to find the value m for any $h > 0$ it suffices to substutute in the obtained estimate the function $u_0(xh)$ as it was done at the end of the proof of Lemma 2. $\square$

Lemma 4. *Let* $p < 0,\ 0 < r \le 1$ *and*

$$m_1 = \sup_h \sup_{y \in K_p(0,h,r)} G[y].$$

Then there is a function $u_1 \in K_p(0, r/2, r)$ *such that* $u_1(x) = u_1(r-x)$, *the function* u_1 *coincides in* $[0, r/2]$ *with the function* u_0 *found in Lemma* 3 *for* $h = r/2$, $G[u_1] = m_1$, *and* $m_1 = 4mr^{2/p-3}$, *where the value of* m *was indicated in Lemma* 3 *for* $h = r/2$.

Proof. Let at first $r = 1$. The existence of the extremal function for any fixed $h \in [0, 1]$ follows from Lemma 3. Furthermore, we can suppose that $u_1(x)$ is monotone and concave in $[0, h]$ and in $[h, 1]$.

By Lemma 3 we have for $y \in K_p(0, h, 1)$ the inequalities

$$h^{3-2/p}\left(\int_0^h |y'(x)|^p dx\right)^{2/p} \leq m \int_0^h y(x)^2 dx,$$

$$(1-h)^{3-2/p}\left(\int_h^1 |y'(x)|^p dx\right)^{2/p} \leq m \int_h^1 y(x)^2 dx,$$

where the value of m, corresponding to $h = 1$, was found in Lemma 3. Let $\int_0^1 |y'(x)|^p dx = 1$ and $\int_0^h |y'(x)|^p dx = a$. Then

$$a^{2/p} h^{3-2/p} + (1-a)^{2/p}(1-h)^{3-2/p} \leq m \int_0^1 y(x)^2 dx.$$

By Lemma 1 the function $F(a, h) = a^{2/p} h^{3-2/p} + (1-a)^{2/p}(1-h)^{3-2/p}$ defined in the square $0 < a < 1$, $0 < h < 1$, has a minimal value $1/4$ at the point $a = h = 1/2$. Therefore,

$$\int_0^1 y(x)^2 dx \geq m/4,$$

i.e. $G[y] \leq 4m$ for all admissible y. On the other hand, if $a = h = 1/2$ and if the function u_1 coincides on $(0, 1/2)$ with the function u_0, found in Lemma 3 for $h = 1/2$, and is even with respect to the point $x = 1/2$, then $G[u_1] = 4m$.

In order to obtain the result for an arbitrary r it suffices to substitute the function $u_1(x)$ by $u_1(xr)$ as it was done at the end of the proof of Lemma 2. $\square$

Lemma 5. *Let $p < 0$ and*

$$m_2 = \sup \sup_{y \in K_p(h)} G[y],$$

where $K_p(h)$ is the class of functions y such that $y \in K_p(0, a, h)$ with some $a \in]0, h[$, $-y \in K_p(h, b, 1)$ with some $b \in]h, 1[$. Then there exists a function $u_2 \in K_p(1/2)$ such that $G[u_2] = m_2$. Moreover,

$$m_2 = 4m_1, u_2(x) = -u_2(1-x),$$

and $u_2(x) = u_1(x)$ for $0 < x < 1/2$, where m_1 and u_1 were found in Lemma 4 for $r = 1/2$.

Proof. The existence of the extremal function for any fixed $h \in [0,1]$ follows from Lemma 4. Furthermore, we can suppose that $u_2(x)$ is monotone and concave in $[0,h/2]$, $[h/2,h]$, $[h,(1+h)/2]$ and in $[(1+h)/2,1]$.

By Lemma 4 we have for $y \in K_p(h)$ the inequalities

$$h^{3-2/p}\Big(\int_0^h |y'(x)|^p dx\Big)^{2/p} \le m_1 \int_0^h y(x)^2 dx,$$

$$(1-h)^{3-2/p}\Big(\int_h^1 |y'(x)|^p dx\Big)^{2/p} \le m_1 \int_h^1 y(x)^2 dx,$$

where the value of m_1, corresponding to $h=1$, was found in Lemma 4. Let $\int_0^1 |y'(x)|^p dx = 1$ and $\int_0^h |y'(x)|^p dx = a$. Then

$$a^{2/p}h^{3-2/p} + (1-a)^{2/p}(1-h)^{3-2/p} \le m_1 \int_0^1 y(x)^2 dx.$$

By Lemma 1 the function $F(a,h) = a^{2/p}h^{3-2/p} + (1-a)^{2/p}(1-h)^{3-2/p}$ defined in the square $0<a<1,\ 0<h<1$ has the minimal value $1/4$ at the point $a=h=1/2$. Therefore,

$$\int_0^1 y(x)^2 dx \ge m_1/4,$$

i.e. $G[y] \le 4m_1$ for all admissible y. On the other hand, if $a=h=1/2$ and if the function u_2 coincides on $(0,1/2)$ with the function u_1, found in Lemma 4 for $h=1/2$, and is odd with respect to the point $x=1/2$, then $G[u_2] = 4m_1 = m_2$. □

Lemma 6. *Let $Q(x)$ be a smooth positive function on $[0,d[$, such that*

$$\lim_{x\to d} Q(x)(d-x)^{-\gamma} = a \ne 0,$$

where $1<\gamma<2$. Let $y(x)$ be a solution of the equation

$$Q(x)y''(x) + my(x) = 0,\ 0<x<d,$$

such that

$$\int_0^d Q(x)y''(x)^2 dx < \infty.$$

Then

$$\lim_{x\to d^-} y(x) = 0,\ \lim_{x\to d^-} y(x)y'(x) = 0.$$

Moreover, we have as $x \to d^-$

$$y(x) = C[d - x + c_1(d - x)^{3-\gamma}(1 + o(1))],$$

$$y'(x) = C[-1 + c_1(3 - \gamma)(d - x)^{2-\gamma}(1 + o(1))],$$
$$y''(x) = Cc_1(3 - \gamma)(2 - \gamma)(d - x)^{1-\gamma}(1 + o(1)).$$

Proof. Set $I[u] = \int_0^d Q(x)u''(x)^2 dx$.

Solving the Cauchy problem we can find two linearly independent solutions $y(x)$ and $z(x)$ such that as $x \to d-$

$$y(x) = d - x + c_1(d - x)^{3-\gamma} + \ldots,\ c_1 = \frac{m}{a(3 - \gamma)(2 - \gamma)};$$

$$z(x) = 1 + c_2(d - x)^{2-\gamma} + \ldots,\ c_2 = \frac{m}{a(2 - \gamma)(1 - \gamma)}.$$

However $z''(x) = (d-x)^{-\gamma}(m/a+o(1))$ and the integral $I[z] = \int_0^d Q(x)z''(x)^2 dx$ is divergent. So the solution with a finite value of I is proportional to y and vanishes at $x = d$. $\square$.

Lemma 7. *Let a function f be summable on $(0, 1)$, $f(x) \geq 0$. Then there are two points a and b such that $0 < a < b < 1$ and a function $y \in C^1[0, 1]$ such that $y'(x)$ is absolutely continuous,*

$$y(0) = y(1) = y'(0) = y'(1) = 0,$$

$y''(x) = f(x)$ if $0 < x < a$ or $b < x < 1$, and $y''(x) = -f(x)$ if $a < x < b$.

Proof. Let at first $f(x) > 0$. Set

$$F(x) = \int_0^x f(t)dt,\ G(x) = \int_0^x F(t)dt,$$

and

$$y'(x) = \begin{cases} F(x), & if\ 0 < x < a, \\ 2F(a) - F(x), & if\ a < x < b, \\ F(x) - F(1), & if\ b < x < 1, \end{cases}$$

where the values of a and b will be indicated in what follows. This function is continuous at the point $x = a$ and at the point $x = b$, if

$$2F(a) + F(1) = 2F(b). \tag{4}$$

Let

$$y(x) = \begin{cases} G(x), & if\ 0 < x < a, \\ 2F(a)(x-a) - G(x) + 2G(a), & if\ a < x < b, \\ G(x) - G(1) - (x-1)F(1), & if\ b < x < 1. \end{cases}$$

This function is continuous at the points $x = a$ and $x = b$, if

$$2F(a)(b-a) - G(b) + 2G(a) = G(b) - G(1) - (b-1)F(1).$$

Set

$$H(x) = xF(x) - G(x).$$

Then

$$H'(x) = xf(x) \geq 0,\ H(0) = 0,$$

and condition (4) implies that

$$2H(a) + H(1) = 2H(b).$$

The points a and b can be found in the following way. Set

$$K(x) = F(x) - F(1)H(x)/H(1) = H(x)[\frac{F(x)}{H(x)} - \frac{F(1)}{H(1)}].$$

Since

$$(F(x)/H(x))' = -F'(x)G(x)/H(x)^2,$$

we see that $K(x) \geq 0,\ K(0) = 0,\ K(1) = 0$, and

$$K(a) - K(b) = -1/2K(1) = 0.$$

Let the function $\rho(t)$ be defined by the equality

$$F(\rho(t)) - 1/2F(1) = F(t).$$

Then $\rho(0) = \xi$, where ξ is a point such that $F(\xi) = F(1)/2,\ 0 < \xi < 1$, and $\rho(\xi) = 1$. The function

$$S(t) = K(t) - K(\rho(t))$$

is such that $S(0) = K(0) - K(\xi) \leq 0$ and $S(\xi) = K(\xi) - K(1) \geq 0$. Therefore, there is a point $a \in (0, \xi)$ such that $S(a) = 0$. If we set $b = \rho(a)$, then we obtain (4) and the equality $K(a) = K(b)$ will also be satisfied.

If $f(x) \geq 0$, then we can construct the function y_ε corresponding to the function $f(x) + \varepsilon$ and pass to limit, which is easy. $\square$

Lemma 8. *Let $0 < \alpha < 1$ and $Q_0(x) = |u_2'(x)|^{2/(\alpha-1)}$, where u_2 is the function found in Theorem 5. Let*

$$m_0 = \inf_{y \in H_0^2(0,1)} \frac{\int_0^1 Q_0(x) y''(x)^2 dx}{\int_0^1 y'(x)^2 dx}.$$

Then $m_0 = M_\alpha$, where M_α was indicated in Theorem 2. The minimal value is attained on the function $\int_0^x u_2(t)dt$.

Proof. Consider the minimizing sequence $y_k(x)$ such that $\int_0^1 y_k'(x)^2 dx = 1$. The integrals $\int_0^{1/4-\varepsilon} y_k''(x)^2 dx$, $\int_{1/4+\varepsilon}^{3/4-\varepsilon} y_k''(x)^2 dx$ and $\int_{3/4+\varepsilon}^1 y_k''(x)^2 dx$ are bounded and one can choose a subsequence converging almost everywhere in $(0, 1/4-\varepsilon)$, $(1/4+\varepsilon, 3/4-\varepsilon)$ and $(3/4+\varepsilon, 1)$ in H^1 and weakly in $H^2(0, 1/4-\varepsilon)$, $H^2(1/4+\varepsilon, 3/4-\varepsilon)$ and $H^2(3/4+\varepsilon, 1)$ for any $\varepsilon \in]0, 1/8[$. Using the diagonalization, one can find a subsequence converging almost everywhere in $(0,1)$ to $v(x)$.

Let us show that

$$\int_{1/4-\varepsilon}^{1/4+\varepsilon} y_k'(x)^2 dx + \int_{3/4-\varepsilon}^{3/4+\varepsilon} y_k'(x)^2 dx \leq C\varepsilon^{2-\gamma} \tag{5}$$

with a constant C independent of ε and k. Indeed, using the equality

$$y_k'(1/4-\varepsilon) = \int_0^{1/4-\varepsilon} y_k''(x) dx,$$

we see that for small $\varepsilon > 0$

$$y_k'(1/4-\varepsilon)^2 \leq \int_0^{1/4-\varepsilon} Q(x) y_k''(x)^2 dx \int_0^{1/4-\varepsilon} Q(x)^{-1} dx \leq C_1 \varepsilon^{1-\gamma}.$$

Since $\int_{1/4+\varepsilon}^{3/4-\varepsilon} y_k'(x)^2 dx < 1$, there exists a $\theta_k \in]1/4+\varepsilon, 3/4-\varepsilon[$ such that $y_k'(\theta_k)^2 \leq 3$ and

$$y_k'(1/4+\varepsilon)^2 \leq 6 + 2\int_{\theta_k}^{1/4+\varepsilon} Q(x) y_k''(x)^2 dx \int_{\theta_k}^{1/4+\varepsilon} Q(x)^{-1} dx \leq C_2 \varepsilon^{1-\gamma}.$$

Analogously, we have

$$y_k'(3/4-\varepsilon)^2 \le C_2\varepsilon^{1-\gamma},\ y_k'(3/4+\varepsilon)^2 \le C_1\varepsilon^{1-\gamma}.$$

If $1/4-\varepsilon < x < 1/4$, then

$$y_k'(x)^2 \le 2y_k'(1/4-\varepsilon)^2$$

$$+2\int_{1/4-\varepsilon}^{x} Q(x)y_k''(x)^2dx \int_{1/4-\varepsilon}^{x} Q(x)^{-1}dx \le C_3(1/4-x)^{1-\gamma}$$

and therefore,

$$\int_{1/4-\varepsilon}^{1/4} y_k'(x)^2dx \le C_3\varepsilon^{2-\gamma}.$$

So (5) is valid. Since we can assume that y_k' converges uniformly to v' outside of the ε-neighbourhood of the points $1/4$ and $3/4$ we see that

$$\int_0^1 v'(x)^2dx = 1,\ \int_0^1 Q(x)v''(x)^2dx \le m_0.$$

However, m_0 is the minimal possible value of the latter integral. Therefore, $\int_0^1 Q(x)v''(x)^2dx = m_0$. The function v satisfies the equation

$$(Q(x)v''(x))'' + m_0v''(x) = 0,$$

$v(0)=0,\ v'(0)=0,\ v(1)=0,\ v'(1)=0$. The function $z_1(x) = v(x)+v(1-x)$ is also minimizing, if it does not vanish identically.

If the function $z_1(x)$ does not vanish identically, then it is even and

$$Q(x)z_1''(x) + m_0z_1 = C.$$

Using Lemma 6 we see that $C = m_0z_1(1/4) = m_0z_1(3/4)$. Set $u(x) = z_1(x) - z_1(1/4)$. Then

$$Q(x)u'' + m_0u = 0,\ u(1/4) = u(3/4) = 0.$$

On the other hand, if $v(x) = w(x) - w(1/4)$, where $w(x) = \int_0^x u_2(t)dt$ and u_2 is the function indicated in Lemma 5, then

$$Q(x)v'' + mv = 0,\ v(1/4) = v(3/4) = 0$$

and $v > 0$ in $(1/4, 3/4)$. If u vanishes in $(1/4, 3/4)$, then by the Sturm theorem, $m_0 > m$, what is impossible. If u does not vanish in $(1/4, 3/4)$, then we obtain using Lemma 6 that

$$(m_0 - m)\int_{1/4}^{3/4} \frac{uv}{Q} dx = (u'v - uv')|_{x=1/4}^{x=3/4} = 0$$

and $m_0 = m$.

If $z_1(x) \equiv 0$, then v is odd and

$$Q(x)v''(x) + m_0 v = A(x - 1/2),$$

where A is a constant. Applying Lemma 6, we see that $A = -4m_0 v(1/4)$. Setting $u(x) = v(x) - A(x - 1/2)/m_0$, we obtain that

$$Qu'' + m_0 u = 0.$$

Moreover, $u(0) = u(1/4) = u(1/2) = u(3/4) = 0$. Applying once again the Sturm theorem, we see that $m_0 > m$, which is impossible. □

Proof.

1. Let Q be a function from the class R_α. According to Lemma 7[4], we can construct a function $y(x)$ such that

$$y''(x)^2 = [Q(x) + 1]^{\alpha-1},\ y(0) = y'(0) = y(1) = y'(1) = 0.$$

Moreover, $y'(x) \geq 0$ for $0 \leq x \leq r$ and $y'(x) \leq 0$ for $r \leq x \leq 1$, so that r is the maximum point of the function $y(x)$.

By our construction

$$\int_0^1 |y''(x)|^p dx = \int_0^1 [Q(x) + 1]^\alpha dx,$$

where $p = 2\alpha/(\alpha - 1)$. The function y' satisfies the conditions of Lemma 5. Therefore,

$$L[Q, y] \leq L[Q + 1, y] = \frac{\int_0^1 |y''|^p dx}{\int_0^1 |y'|^2 dx}$$

$$= G[y'](\int_0^1 (Q(x) + 1)^\alpha dx)^{1/\alpha} \leq 2^{1+1/\alpha} G[y'] \leq C$$

[4]Lemmas 1-8 are stated in the following Section

since $(Q+1)^\alpha \leq 2^\alpha(Q^\alpha+1)$, $\int_0^1 Q^\alpha dx = 1$ and $G[y'] \leq m_2$ by Lemma 5. Thus $M_\alpha \leq C$.

2. On the other hand, $\square$

Proof of Theorem. Let us show at first that $m_\alpha = 0$. One can take a function y vanishing in $[0,1/2]$ and such that $\int_0^1 y'^2 dx = 1$. Since the function Q can have arbitrarily small values in $[1/2,1]$, the value of m_α is equal to 0.

Let Q be an arbitrary positive function, satisfying (1). By Lemma 7 there exists a function $y_Q(x) \in \mathcal{H}$ such that $Q(x) = |y_Q''|^{2/(\alpha-1)}$, and $y_Q''(x) > 0$ in $]0,h[$ and in $]r,1[$, $y_Q''(x) < 0$ in $]h,r[$, so that $L[Q, y_Q] = G[y_Q']$. Therefore, $M_\alpha \leq m_2$, where m_2 is the number found in Lemma 5.

On the other hand, we have the equality $L[Q_0, y_0] = G[y_0']$, if $Q_0(x) = |y_0''(x)|^{2/(\alpha-1)}$ and y_0 is equal to $\int_0^x u_2(t)dt$, where u_2 is the function indicated in Lemma 5. By Lemma 8 we have

$$m = \inf_{y \in H_0^2(0,1)} \frac{\int_0^1 Q_0(x) y''(x)^2 dx}{\int_0^1 y'(x)^2 dx} = L[Q_0, y_0].$$

The proof is complete. $\square$

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