Uncertainty estimates and L_2 bounds for the Kuramoto-Sivashinsky equation

Jared C. Bronski*
Tom Gambill*

September 23, 2005

Abstract

We consider the Kuramoto-Sivashinsky (KS) equation in one dimension with periodic boundary conditions. We apply a Lyapunov function argument similar to the one first introduced by Nicolaenko, Scheurer, and Temam [18], and later improved by Collet, Eckmann, Epstein and Stubbe[1], and Goodman [10] to prove that $\limsup_{t\to\infty} \|u\|_2 \leq CL^{\frac{3}{2}}$. This result is slightly weaker than that recently announced by Giacomelli and Otto [9], but applies in the presence of an additional linear destabilizing term. We further show that for a large class of functions ϕ_x the exponent $\frac{3}{2}$ is the best possible from this line of argument. Further, this result together with a result of Molinet[17] gives an improved estimate for L_2 boundedness of the Kuramoto-Sivashinsky equation in thin rectangular domains in two spatial dimensions.

1 INTRODUCTION

1.1 Background

The Kuramoto-Sivashinsky (KS) equation

$$u_t = -u_{xxxx} - u_{xx} + uu_x \qquad \int u(x,0)dx = 0$$

arises as a model of certain hydrodynamic problems, most notably the propagation of flame fronts[20]. The KS equation is interesting mathematically because the linearization about the zero state

$$u_t = -u_{xxxx} - u_{xx}$$

subject to periodic boundary conditions on $[-L, L]$ has a large number ($O(L/\pi)$) of exponentially growing modes. The growth of these modes corresponds, in the

*Department of Mathematics,University of Illinois Urbana-Champaign, 1409 W. Green St, Urbana IL 61801

combustion problem, to the development of nontrivial structures. In addition to its importance as a model for flame fronts[20] and phase turbulence[14] and plasmas[15] the KS equation has become one of the canonical models for spatio-temporal chaos in 1+1 dimensions[12, 13, 16].

Nicolaenko, Scheurer and Temam[18] gave the first long-time boundedness result for the Kuramoto-Sivashinsky equation, showing that $\limsup_{t\to\infty} \|u\|_2 \leq CL^{\frac{5}{2}}$ for odd initial data, as well as showing that bounds on the L_2 norm imply bounds on the dimension of the attractor. The L_2 estimate was improved by Collet, Eckmann, Epstein and Stubbe[1] who extended it to any mean-zero initial data and improved the exponent from $\frac{5}{2}$ to $\frac{8}{5}$, and by Goodman[10], who extended it to any mean-zero initial data with the same exponent. All of these papers use a version of the original argument of Nicolaenko, Scheurer and Temam, namely to establish that the function $\|u-\phi\|_2^2$ is a Lyapnuov function for an appropriately chosen ϕ and $\|u\|_2$ sufficiently large. There are also two bounds which do not fit into this Lyapunov function framework, that of Ilyashenko[11], and the very recent paper of Otto and Giacomelli[9]. The latter, which treats the KS equation as a perturbation of the Burger's equation, is currently the best estimate, establishing that

$$\limsup_{t\to\infty} \|u\|_2 = o(L^{\frac{3}{2}}).$$

In this paper we give an elementary argument of the Lyapunov function type which establishes the slightly weaker result

$$\limsup_{t\to\infty} \|u\|_2 = O(L^{\frac{3}{2}}).$$

Our proof applies equally to the destabilized Kuramoto-Sivashinsky (dKS) equation:

$$u_t = -u_{xxxx} - u_{xx} + \gamma u + u u_x \qquad \gamma > 0 \qquad \int u(x,0)dx = 0.$$

It was shown by Wittenberg[23] that this equation has stationary solutions which satisfy $\|u\| \propto L^{\frac{3}{2}}$. Since a Lyapunov function argument for the KS equation also applies to the dKS equation (for sufficiently small γ) Wittenberg argued that $\frac{3}{2}$ is the best exponent that one can expect from the Lyapunov function approach. This paper completes this circle of ideas, by showing that this exponent can actually be achieved.

We also give a independent scaling argument that motivates the choice of ϕ_x and shows that for a large class of potentials $\frac{3}{2}$ is the best exponent possible from this line of argument. This argument is useful because it makes clear the physical basis of the scaling, and is potentially applicable to other equations.

1.2 Fundamental Lemmas

We begin by stating two basic lemmas which form the core of the Lyapunov function argument. These lemmas are basically equivalent to equations 2.11,12 and 3.10-3.12 in [18], or analogous results in [1],[10]. It is worth noting that similar ideas of considerably greater generality have been used by Constantin and Doering to establish bounds on energy dissipation in fluids, and generally go by the name 'background flow method.'[2, 3]

Lemma 1. *Given $u = u(x,t) \in L_2[-L,L]$ and $\phi = \phi(x) \in L_2[-L,L]$ satisfying the following inequality:*

$$\tfrac{\partial}{\partial t}\|u-\phi\|_2^2 \leq -\lambda\|u\|_2^2 + M^2 \tag{1}$$

for some constants $\lambda > 0$ and M, then $B(0, R^{**})$, the ball of radius R^{**} centered about the origin, is an attracting region, where the radius R^{**} is given by

$$R^{**} = \sqrt{2\|\phi\|_2^2 + \frac{2M^2}{\lambda}} + \|\phi\|_2. \tag{2}$$

To show this, let $\|u\|_2 \geq R^{**}$ then

$$\|u-\phi\|_2 \geq \|u\|_2 - \|\phi\|_2 \geq \sqrt{2\|\phi\|_2^2 - \frac{2M^2}{\lambda}}$$

The parallelogram law implies

$$-\lambda\|u-\phi\|_2^2 \geq -2\lambda\|u\|_2^2 - 2\lambda\|\phi\|_2^2$$

which in turn gives

$$\tfrac{\partial}{\partial t}\|u-\phi\|_2^2 \leq \lambda\|\phi\|_2^2 + M^2 - \tfrac{\lambda}{2}\|u-\phi\|_2^2.$$

If we apply the obvious Gronwall estimate to the above inequality it is apparent that $B(\phi, R^*)$, the ball of radius R^* centered about ϕ is exponentially attracting, with $R^{*2} = \|\phi\|^2 + \frac{2M^2}{\lambda}$. The triangle inequality implies $B(\phi, R^*) \subset B(0, R^{**})$. ♣

Lemma 2. *For any $\phi \in \dot{H}^2_{\text{per}}$ and $u(x,t)$ solving the Kuramoto Sivashinsky equation we have (after some rescaling) the inequality*

$$\frac{1}{128}\frac{\partial}{\partial t}\int_{-64L}^{64L}(u-16\phi)^2 \leq 8\Big(\int_{-64L}^{64L} u_y^2 - u_{yy}^2 - \tilde{\phi}_y u^2\Big) + \int_{-64L}^{64L} 16\tilde{\phi}_y^2 + 2(16^3)\tilde{\phi}_{yy}^2$$

A straightforward calculation gives

$$\frac{1}{2}\frac{\partial}{\partial t}\|u-\phi\|_2^2 = \int_{-L}^{L} u_t(u-\phi) = \int(-u_{xx} - u_{xxxx} - uu_x)(u-\phi).$$

After integrating by parts and applying periodic boundary conditions this becomes

$$\frac{1}{2}\frac{\partial}{\partial t}\|u-\phi\|_2^2 \quad = \quad \int u_x^2 - u_{xx}^2 - \phi_x u_x + \phi_{xx}u_{xx} - \frac{1}{2}\phi_x u^2.$$

Applying the Cauchy-Schwartz inequality in the form $<f,g>\leq \frac{p}{2}<f,f>+\frac{1}{2p}<g,g>$ gives

$$\frac{1}{2}\frac{\partial}{\partial t}\|u-\phi\|_2^2 \quad \leq \quad \int (1+\frac{1}{2p})u_x^2 + (\frac{1}{2q}-1)u_{xx}^2 + \frac{p}{2}\phi_x^2 + \frac{q}{2}\phi_{xx}^2 - \frac{1}{2}\phi_x u^2.$$

If we then make the substitution $\phi = \gamma\tilde{\phi},\ y = \beta x$, we find that

$$\frac{1}{128}\frac{\partial}{\partial t}\int_{-\beta L}^{\beta L}(u-16\phi)^2 dy \leq \int_{-\beta L}^{\beta L} \frac{1+2p}{2p}\beta u_y^2 + \frac{1-2q}{2q}\beta^3 u_{yy}^2 + \frac{p}{2}\beta\gamma^2\tilde{\phi}_y^2 + \frac{q}{2}\beta^3\gamma^2\tilde{\phi}_{yy}^2 - \frac{1}{2}\gamma\tilde{\phi}_y u^2 dy\ .$$

Finally, taking $p=\frac{1}{2}, q=1, \beta=64, \gamma=16$ we get

$$\frac{1}{128}\frac{\partial}{\partial t}\int_{-64L}^{64L}(u-16\phi)^2 dy \leq 8\int_{-64L}^{64L}(u_y^2 - u_{yy}^2 - \tilde{\phi}_y u^2)dy + \int_{-64L}^{64L} 16\tilde{\phi}_y^2 + 2(16^3)\tilde{\phi}_{yy}^2,$$

as claimed.
♣

Remark 1. *Since we are only concerned with the scaling of the estimates with* L*, and not with the actual constants, we will henceforth drop the tilde and replace* βL *with* L *and* y *with* x*. The preceeding lemmas show that, if we can construct* $\phi \in \dot{H}^2_{per}$ *such that the coercivity estimate*

$$<u, Ku>= \int u_{xx}^2 - u_x^2 + \phi_x u^2 dx > \delta\|u\|_2^2 > 0$$

for some δ *independent of* L*, then we get an estimate of the form*

$$\limsup_{t\to\infty}\|u\|_2 \leq R^{**} = \sqrt{c_1\|\phi\|_2^2 + c_2\|\phi_x\|_2^2 + c_3\|\phi_{xx}\|_2^2} + c_4\|\phi\|_2^2.$$

Since it is clear that that R^{**} *is comparable to the* H^2 *norm,* $c\|\phi\|_{H^2} \leq R^{**} \leq C\|\phi\|_{H^2}$*, we will write this in the form*

$$\limsup_{t\to\infty}\|u\|_2 \leq c\|\phi\|_{H^2}.$$

Our argument proceeds along the same lines as that of previous papers, but with a better construction of the function ϕ_x, which leads to better exponents than in previous Lyapunov function arguments. We first construct a ϕ_x such that the operator K defined above is positive definite for u satisfying Dirichlet boundary condition. This establishes an L_2 bound for odd solutions of the KS equation, which are preserved under the flow. This result can be extended to all mean-zero data by allowing a time-dependent ϕ_x which translates under a kind of gradient-flow dynamics, as was first done by Collet, Eckmann, Epstein and Stubbe. Our construction is done in real space, unlike the constructions of Nicolaenko, Scheurer and Temam, and Collet, Eckmann, Epstein and Stubbe, where the construction uses a clever cancellation in Fourier space. We feel that the real space construction clarifies the role of uncertainty estimates in determining the positivity of the operator, and makes it easier to see what the optimal scaling should be.

2 MAIN RESULTS

2.1 Scaling, Uncertainly, and Bounds

In this section we present our main results. We begin with a discussion of the role of scaling and uncertainty in determining exponents in the Lyapunov function approach to proving boundedness of the KS equation. The construction of a suitable function ϕ_x can be viewed as a competition between kinetic energy and potential energy terms in the operator, and relatively simple scaling arguments make it clear how the function should scale with L, the length of the interval. With this as motivation, we proceed to prove that a suitable Lyapunov function with the critical scaling exponents can be constructed. The main technical tools will be a Hardy-type inequality, which allows us to derive a lower bound on a second order kinetic energy term with a Dirichlet boundary condition in terms of a standard first-order kinetic energy term, together with a lower bound on a Schrodinger operator in terms of a finite dimensional quadratic form.

As outlined in the previous section the basic strategy is to choose a periodic function ϕ_x of zero mean such that the following quadratic form is coercive,

$$<u, Ku>= \int u_{xx}^2 - u_x^2 + \phi_x u^2 \geq \delta \|u\|^2,$$

for u satisfying Dirichlet boundary conditions and some positive δ independent of L. Doing so gives a bound on the radius of the attracting ball in L_2 (for odd solutions) which scales like $\|\phi\|_{H^2}$. A sketch of the strategy which has previously been followed for constructing such a potential[18, 1] is this: One constructs a potential in Fourier space which is constant for a large range of wavenumbers, and decaying thereafter. For such a potential the quadratic form above looks like a diagonal piece plus a piece which is supported only at very large wavenumbers. One then uses the rapid growth of the dispersion relation $k^4 - k^2$ to show diagonal dominance and thus positivity of the operator. The real space translation of this strategy is as follows: One constructs a potential that is large and negative on a small set near the origin, and positive on the rest of the interval, in such a way that the potential is mean zero. The uncertainty principle, together with the Dirichlet boundary condition, implies that little of the mass of the ground state can be concentrated in the small region where the potential is negative, so that the net effect is to produce a positive ground state eigenvalue. In this section we present a scaling argument which suggests the best possible estimate one can get of this form. In the next section we show that this estimate can actually be achieved.

We will assume for simplicity of discussion that for $x \in (-L, L)$ the function ϕ_x takes the following form,

$$\phi_x = \gamma L^{c_2 - c_1 - 1} + L^{c_2} \tilde{q}(x L^{c_1}) \qquad \gamma, c_{1,2} > 0$$

where γ is a constant and $\tilde{q}$ is compactly supported C^2 function, with ϕ_x extended to a $2L$ periodic function in the usual way. The functions constructed by

Goodman and in the current paper are exactly of this form, while the functions ϕ_x constructed by other authors can all be written as a sum of a constant and a function which is rapidly decaying away from the origin, though not necessarily compactly supported. The present discussion can be extended to such potentials under some very mild technical assumptions. For details see the Ph.D. thesis of one of the authors[8]. For reference the functions constructed by Temam et. al and by Goodman have scaling exponents $c_1 = 1, c_2 = 2$ while the function constructed by Collet et. al. has scaling exponents $c_1 = \frac{2}{5}, c_2 = \frac{7}{5}$.

Our first observation is that the operator cannot be positive for $c_2 - c_1 - 1 < 0$. This follows from a straightforward test-function argument using a delocalized test function such as $u = L^{-\frac{1}{2}} \sin(\frac{k\pi x}{L})$, for suitably chosen k. Next, if one makes the rescaling $y = L^{c_1} x$, the quadratic form becomes

$$L^{3c_1} \left(\int_{-L^{1+c_1}}^{L^{1+c_1}} u_{yy}^2 - L^{-2c_1} u_y^2 + L^{c_2 - 4c_1} \tilde{q}(y) u^2(y) dy \right) + \gamma L^{c_2 - c_1 - 1} \|u\|_2^2.$$

Note the prefactor of $L^{c_2 - 4c_1}$ in front of the potential $\tilde{q}$. Motivated by this, we refer to potentials for which $c_2 - 4c_1 < 0$ as weak potentials, those for which $c_2 - 4c_1 > 0$ as strong potentials, and those for which $c_2 = 4c_1$ as critical potentials. Strong potentials are those for which the potential energy term dominates the low-lying eigenvalues in the limit $L \to \infty$, while weak potentials are those for which the kinetic energy term dominates. All of the potentials constructed in previous papers are weak potentials, with the potential in [1] being closest to critical.

It is clear from another simple test-function argument, this time with a localized test-function, that in the case of a strong potential the operator K again cannot be positive. Simply taking a compactly supported test function whose support is contained in a region where $\tilde{q} < 0$ gives an estimate of the following form

$$\lambda_0(K) \leq -C L^{c_2 - c_1} + O(L^{3c_1}, L^{c_1}, L^{c_2 - c_1 - 1}).$$

where the three error terms come from the u_{xx}^2, u_x^2 and u^2 terms respectively. A simple calculation shows that the H^2 norm of the potential ϕ_x is bounded below by

$$\|\phi\|_{H^2} \geq \|\phi_{xx}\|_2 = O(L^{c_2 + \frac{c_1}{2}}).$$

Thus the best estimate possible for a ϕ_x of this form is given by the solution to the constrained minimization problem

$$\begin{aligned} \text{minimize} \quad c_2 \quad &+ \quad \frac{c_1}{2} \quad \text{subject to} && (3) \\ c_2 \quad &\geq \quad c_1 + 1 && (4) \\ c_2 \quad &\leq \quad 4c_1 && (5) \end{aligned}$$

It is easy to check that the solution to this constrained minimization problem is given by

$$c_2 = \frac{4}{3} \qquad c_1 = \frac{1}{3}.$$

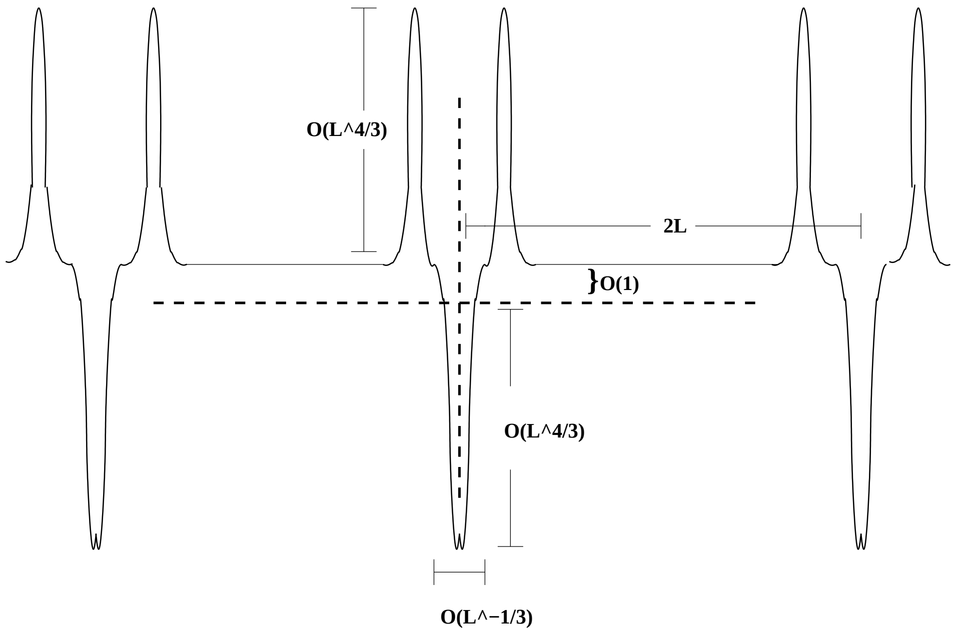

Figure 1: A cartoon of the function ϕ_x constructed in this paper.

A cartoon of such a potential is shown in figure(1). A potential of this form would give an estimate of the radius of the attractor that scales like $L^{\frac{4}{3}+\frac{1}{2}\cdot\frac{1}{3}} = L^{\frac{3}{2}}$. We show in this paper that these critical exponents can actually be achieved.

It is worth noting that the scaling given above for the potential function ϕ_x is exactly the same as that for the viscous shock profile for the destabilized KS equation constructed by Wittenberg[23]. This is not too surprising, since the same Lyapunov function argument applies to the destabilized KS equation, and one might reasonably expect that the Lyapunov function should look something like the steady state. In fact, it is easy to check that if ϕ_γ is a stationary solution to the destabilized KS equation, and the linearized operator is negative semi-definite, then $\|u-\phi\|^2$ is a Lyapunov function for sufficiently large $\|u\|$.

This calculation makes clear the construction of a Lyapunov function for this problem involves a competition between the kinetic energy terms in the functional (which scale with the width of the potential) and the potential energy terms, which scale with the height of the potential. In particular it should be clear from this calculation that the scaling depends crucially on the order of the operator. In particular one expects *different* critical exponents for a second order operator, since the kinetic energy term is less effective at small scales than the analogous term for a fourth order operator. Along these lines we make a couple of other comments, directed specifically at the papers of Nicolaenko, Scheurer and Temam and of Goodman. In [18] and [10] they make an additional simplification by using the inequality $\partial_{xxxx} + \partial_{xx} \geq -\partial_{xx} + 1$ to bound the operator K from below by a standard second order Schrodinger operator $\tilde{K}$:

$$<u, \tilde{K}u> = \int u_x^2 - u^2 + \phi_x u^2 \;\leq\; <u, Ku> = \int u_{xx}^2 - u_x^2 + \phi_x u^2$$

If one carries out the same scaling analysis presented above for this quadratic form one finds that the exponents c_1, c_2 defined above must satisfy the inequalities

$$c_2 \geq c_1 + 1 \tag{6}$$

$$c_2 \leq 2c_1, \tag{7}$$

along with the same estimate of the penalty term due to the potential:

$$\|\phi\|_{H^2} \geq \|\phi_{xx}\|_2 = O(L^{c_2 + \frac{c_1}{2}}).$$

Carrying out this minimization problem gives the critical exponents $c_1 = 1, c_2 = 2$, giving an estimate of the radius of the attracting ball of $R = CL^{\frac{5}{2}}$. Thus the estimates in [18] and [10] are the best possible for potentials of this form and the second order operator $\tilde{K}$.

This calculation is, in a sense, complementary to Lieb-Thirring type inequalities. In Lieb-Thirring inequalities one attempts to maximize some measure of the negative part of the spectrum of an operator over all potentials with a fixed norm. In the Lyapunov function method one would like to minimize the negative part of the spectrum, to obtain a positive operator. Unfortunately, there appear to be only few results of this sort for operators of higher order than second (see, however, the work of Tadjbakhsh and Keller[22] and Cox and Overton[4] on the optimal shape of columns).

As a sidenote we remark that the Burgers-Sivashinsky equation, being second order, provides an amusing application of this point of view. An analysis of the Burgers-Sivashinsky (BS) equation

$$u_t = u_{xx} + u + uu_x$$

by the Lyapunov function method leads to a bound of the form

$$\frac{d}{dt}\|u - \phi\|^2 \leq -\lambda_0 \|u\|^2 + c\|\phi_x\|_2^2$$

where λ_0 is the smallest eigenvalue of the operator

$$Hu = -u_{xx} + \phi_x u$$

One can choose the potential ϕ_x in an 'optimal' way by maximizing the ground state eigenvalue $\lambda_0(\phi_x)$ as a functional of ϕ_x subject to the constraints $\int \phi_x^2 dx =$

constant. Formally one has

$$\max_{\phi_x} \lambda_0(\phi_x)\|u\|_2^2 - \mu \int \phi_x^2 \quad = \tag{8}$$

$$\max_{\phi_x} \min_u \int u_x^2 + \phi_x u^2 - \mu \phi_x^2 \quad = \tag{9}$$

$$\min_u \max_{\phi_x} \int u_x^2 + \phi_x u^2 - \mu \phi_x^2 \tag{10}$$

where μ is the Lagrange multipliers associated with the constraint. Of course the replacement of the max min by min max needs justification, but this can be done fairly easily in the second order case. Solving the maximization over ϕ_x leads to $\phi_x = \frac{1}{2\mu}\left(u^2 - \|u\|_2^2/(2L)\right)$ and thus to the functional

$$\min_u \int u_x^2 + \frac{1}{4\mu}(u^2 - \|u\|_2^2/(2L))^2$$

The solution of this minimization problem can be expressed in closed form in terms of Jacobi elliptic functions[8].

2.2 Proof of Main Results

In this section we show that the critical exponents $c_1 = \frac{1}{3}, c_2 = \frac{4}{3}$ given by the solution to Eqs (3,4,5) can actually be achieved. Our main techniques are a higher-order analog of the Hardy inequality together with an elementary uncertainty estimate, which allow us to bound the quadratic form from below by a finite-dimensional one. Our main result can be stated as follows

Theorem 1. *$\exists$ a $2L$ periodic potential function ϕ_x such that*

$$\int u_{xx}^2 - u_x^2 + \phi_x u^2 \geq \frac{1}{4} \int u_{xx}^2 + u^2$$

$\forall u \in C^2[-L, L]$ with $u(0) = 0$. Further we have the estimate $\|\phi\|_{H^2} \leq cL^{\frac{3}{2}}$.

Note: The above theorem requires only a single Dirichlet boundary condition at the origin, and thus applies to many different densely defined domains for the operator $K = \partial_{xxxx} + \partial_{xx} + \phi_x$. We will primarily be interested in the domain of odd, periodic flows, which is preserved under the KS flow.

The proof of this theorem is presented as a series of simple lemmas. The first lemma is an uncertainty inequality that allows us to estimate the second order kinetic energy term by a first order kinetic energy which is more analytically tractable. This lemma has an advantage over the standard Poincare inequality, since it is in some sense 'local' and does not scale badly with large intervals. The downside is that we 'use up' the Dirichlet boundary condition, so we are forced to consider a larger domain for v.

Lemma 3. *Suppose that $u \in C^\infty$ with $u(0) = 0$. Then, if $v(y) = \frac{u(y)}{y}$ we have the inequality*

$$\int_{-a}^{a} \frac{1}{4} u_{yy}^2(y) \geq \int_{-a}^{a} \frac{1}{2} \left(\frac{u(y)}{y}\right)_y^2 = \int_{-a}^{a} \frac{1}{2} v_y^2 dy$$

Proof: Since $u(y) = yv(y)$ we have $u_{yy} = yv_{yy} + 2v_y$ and after substitution into the above integral we get

$$\begin{aligned} \int_{-a}^{a} u_{yy}^2 dy &= \int_{-a}^{a} (yv_{yy} + 2v_y)^2 dy \\ &= \int_{-a}^{a} y^2 v_{yy}^2 dy + 2\int_{-a}^{a} v_y^2 dy + 2a\left(v^2(a) + v^2(-a)\right) \geq 2\int_{-a}^{a} v_y^2. \end{aligned}$$

Here we have integrated by parts once and used the fact that the term $2yv_y^2|_{-a}^{a} = 2a(v^2(a) + v^2(-a))$ is positive.
♣

Remark 2. *This is essentially a higher order analog of the Hardy inequality, which says that for $F \in C^1, F(0) = 0$ one has the estimate*

$$\int |F_x|^2 dx \geq \frac{1}{4} \int \frac{|F(x)|^2}{x^2} dx$$

In the next lemma we show the positivity of an operator which we will later use to bound then operator K from below.

Lemma 4. *Define a piecewise constant compactly supported function $Q(y)$ as follows:*

$$Q(y) = \begin{cases} -q_0, & \text{when } 0 \leq |y| \leq \frac{a}{2} \\ q_1 & \text{when } \frac{a}{2} < |y| \leq a \\ 0 & \text{when } a < |y| \end{cases},$$

with a, q_0, q_1 positive constants satisfying the inequalities

$$q_0 a^2 < 1 \tag{11}$$

$$q_1 > \frac{q_0}{1 - a^2 q_0} \tag{12}$$

Then for all $v \in H^1$ we have

$$\int \frac{1}{2} v_y^2 + Q(y) v^2 dy > 0 \ .$$

Proof: We will show that $\int_0^a \frac{1}{2} v_y^2 + Q(y)v^2 dy > 0$ for any $v \in H^1$. Since Q is even the same argument holds for $\int_{-a}^{0} \frac{1}{2} v_y^2 + Q(y)v^2 dy$. Obviously the integral over $|y| > a$ is positive (since Q is zero) and can thus be neglected. Note that we are not assuming any particular boundary conditions on v.

For any $v \in H^1$ and any two points y_1 and y_2 we have the elementary uncertainty inequality

$$\int_{y_1}^{y_2} v_y^2 \geq \frac{(v(y_1) - v(y_2))^2}{y_2 - y_1} .$$

Since $v \in H^1$ is continuous we can 'sample' $v(y)$ at three location $y_0 = 0,\ y_1 \in (0, a/2), y_2 \in (a/2, a)$ defined as follows

$$\begin{aligned} v(y_0) &= v_0 = v(0) \\ v(y_1) &= v_1 = \max_{y \in (0,a/2)} |v(y)| \\ v(y_2) &= v_2 = \min_{y \in (a/2,a)} |v(y)|. \end{aligned}$$

In the case where there is not a unique point in $(0, a/2)$ at which $|v|$ attains its maximum y_1 can be chosen to be any point at which the maximum is achieved, and similarly for y_2.

One has the obvious lower bound on the kinetic energy in terms of the v_i, y_i:

$$\begin{aligned} \int_0^a \frac{1}{2} v_y^2 &\geq \int_0^{y_1} \frac{1}{2} v_y^2 + \int_{y_1}^{y_2} \frac{1}{2} v_y^2 \\ &\geq \frac{(v_1 - v_0)^2}{a} + \frac{(v_2 - v_1)^2}{2a}, \end{aligned}$$

as well as a bound on the potential energy term,

$$\begin{aligned} \int_0^a Q(y) v^2 &= \int_0^{\frac{a}{2}} Q(y) v^2 + \int_{\frac{a}{2}}^a Q(y) v^2 \\ &\geq \frac{-q_0 v_1^2 a}{2} + \frac{q_1 v_2^2 a}{2}. \end{aligned}$$

The kinetic energy bound is clearly not sharp, and can be improved, but it suffices to prove the lemma. Combining these two lower bounds we find that the functional is bounded below by a quadratic form in three unknowns,$v_{0,1,2}$:

$$\int_{-a}^a \frac{1}{2} v_y^2 + Q(y) v^2 \quad \geq \quad \frac{(v_1 - v_0)^2}{a} + \frac{(v_2 - v_1)^2}{2a} - \frac{q_0 v_1^2 a}{2} + \frac{q_1 v_2^2 a}{2}.$$

The quadratic form is given by $v^T \mathbf{A} v$, where $\mathbf{A}$ is defined by,

$$\mathbf{A} \equiv \begin{bmatrix} \frac{1}{a} & -\frac{1}{a} & 0 \\ -\frac{1}{a} & \frac{3}{2a} - \frac{a q_0}{2} & -\frac{1}{2a} \\ 0 & -\frac{1}{2a} & \frac{1}{2a} + \frac{a q_1}{2} \end{bmatrix}$$

A symmetric matrix $\mathbf{A}$ is positive definite if all of the principle minors are positive, and therefore $v^T \mathbf{A} v > 0$ when the following two inequalities are satisfied.

$$q_0 < \frac{1}{a^2} \tag{13}$$

$$q_1 - q_0 - a^2 q_0 q_1 > 0. \tag{14}$$

For a fixed $a > 0$ the above inequalities always have a solution in the positive quadrant of the (q_0, q_1) plane above the hyperbola defined by $q_1 - q_0 - a^2 q_0 q_1 = 0$.For purposes of this paper we can choose any constants q_0, q_1, a fixed (independent of L) such that the above conditions are satified.
♣

Next, we show that we can construct a modified potential $\tilde{Q}$ such that $\tilde{q}(y) = \tilde{Q}(y)/y^2$ is smooth and the quadratic form does not decrease. This is the content of the next lemma.

Lemma 5. *For any constant* μ *there exists a potential function* $\tilde{Q}$ *such that*

- $\tilde{q} = \tilde{Q}(y)/y^2 \in C_0^\infty$ *and*
- $\int \tilde{q} \leq -\mu$
- $\int \frac{1}{2} v_y^2 + \tilde{Q} v^2 \geq 0$

Proof: In the standard way we define $f(y)$ to be a non-decreasing C^∞ function satisfying

$$f(y) = \begin{cases} 0 & y \leq 0 \\ 1 & y \geq 1 \end{cases}$$

Clearly we have $\lim_{y\to 0} f^{(n)}(y) = 0$ and thus $\lim_{y\to 0} y^{-k} f(y) = 0$ Define $\tilde{Q}$ to be even $\tilde{Q}(y) = \tilde{Q}(-y)$ and defined for $y \geq 0$ by

$$\tilde{Q}(y) = \begin{cases} -q_0 f(\frac{y}{\delta}) & y \in (0, \delta) \\ -q_0 & y \in (\delta, \frac{a}{2} - \delta) \\ -q_0 + (q_0 + q_1) f(\frac{y - \frac{a}{2} + \delta}{\delta}) & y \in (\frac{a}{2} - \delta, \frac{a}{2}) \\ q_1 & y \in (\frac{a}{2}, a) \\ q_1 f(1 + \frac{a-y}{\delta}) & y \in (a, a + \delta), \end{cases}$$

where δ is small and will be chosen later. Clearly $\tilde{q} = \tilde{Q}/y^2$ is in C_0^∞, and $\tilde{q}(0) = 0$. It is easy to check that $\int \tilde{q} \approx -\frac{q_0}{\delta}$ for δ small, and thus the mean $\int \tilde{q}$ can be made arbitrarily negative. Finally note that $\tilde{Q} \geq Q$ and thus $\int \frac{1}{2} v_y^2 + \tilde{Q} v^2 \geq 0$.
♣

We are now in a position to prove our main theorem, in which we construct our potential ϕ_x. The Cauchy-Schwartz inequality implies that $-\int u_x^2 \geq -\frac{1}{2} \int u_{xx}^2 - \frac{1}{2} \int u^2$, so it clearly suffices to show that

$$\int \frac{1}{4} u_{xx}^2 + (\phi_x - \frac{3}{4}) u^2 \geq 0$$

We write the mean zero function ϕ_x in the form

$$\phi_x = q(x) - \langle q \rangle$$

where $\langle \cdot \rangle$ denotes the mean value on $[-L, L]$:

$$\langle q \rangle = \frac{1}{2L} \int_{-L}^{L} q(x) dx.$$

If we can choose q such that $<q>\leq -\frac{3}{4}$ and

$$\int \frac{1}{4}u_{xx}^2 + q(x)u^2 \geq 0$$

then we are done.

Based on the scaling arguments of the previous section we define $q(x)$ to be

$$q(x) = L^{\frac{4}{3}}\tilde{q}(xL^{\frac{1}{3}}) \qquad x \in [-L, L]$$

where $\tilde{q}$ is compactly supported. Obviously this can be extended to a periodic function in the standard way. After the rescaling $y = L^{\frac{1}{3}}x$ the quadratic form in equation (2.2) becomes

$$L\int_{-L^{\frac{4}{3}}}^{-L^{\frac{4}{3}}} \frac{1}{4}u_{yy}^2 + \tilde{q}(y)u^2 dy.$$

From lemma 3 we have the inequality

$$\int \frac{1}{4}u_{yy}^2 + \tilde{q}(y)u^2 dy \geq \int \frac{1}{2}v_y^2 + y^2\tilde{q}(y)v^2 dy = \int \frac{1}{2}v_y^2 + \tilde{Q}(y)v^2 dy$$

From the results of lemma 5 we can choose $\tilde{Q}$ in such a way that the the above is positive for all $u \in C_0^\infty$ with $u(0) = 0$. Since this is a dense domain the operator is positive under Dirichlet boundary conditions.

Remark 3. *The fact that $\tilde{q}$ can be chosen to have arbitrarily negative mean and still generate a positive operator K implies that ϕ_x can be chosen such that $< u, Ku >\geq \delta\|u\|^2$ for any constant δ independent of L with a bound $\|\phi\|_{H^2} \leq CL^{\frac{3}{2}}$, where C depends on δ but is independent of L. Thus the above argument gives an L_2 of the destabilized KS equation*

$$u_t = -u_{xxxx} - u_{xx} + \gamma u + uu_x$$

which scales like $L^{\frac{3}{2}}$ for any fixed γ.

We have constructed a potential function ϕ_x such that the operator K is bounded below on the set of functions satisfying a Dirichlet boundary condition. All that remains to be checked is that the H^2 norm of the potential scales correctly with L. This is the content of the next lemma.

Lemma 6. *The potential ϕ satisfies $\|\phi\|_{H^2} \leq CL^{\frac{3}{2}}$.*

Proof: From the definition

$$\phi_x =<q> -q(x) =<q> -L^{\frac{4}{3}}\tilde{q}(xL^{\frac{1}{3}})$$

it is clear on rescaling that $\|\phi_x\|_2^2 = O(L^{\frac{7}{3}})$ and that $\|\phi_{xx}\|_2^2 = O(L^3)$. Thus we only need to estimate $\|\phi\|_2^2$. Since ϕ_x is defined to be

$$\phi_x(x) =<q> -q(x)$$

we have

$$\phi(x) = \int_0^x \phi_s ds = \int_0^x q(s) - <q> ds,$$

and after the substitution $y = sL^{1/3}$ this becomes

$$\phi(x) = L \int_0^{xL^{1/3}} \tilde{q} dy - <q> x.$$

We have the obvious estimate

$$|\phi(x)| \leq L \int_0^{xL^{1/3}} |\tilde{q}(y)| dy + <q> L$$

Since $\tilde{q}$ is bounded (independently of L) and supported on $[-a, a]$ we have

$$|\phi(x)| \leq caL + <q> L = O(L)$$

The L_2 bound now follows since

$$\|\phi\|_2^2 \leq (2L)\|\phi\|_\infty^2 = O(L^3)$$

♣

2.3 Extension to Arbitrary Initial Data

The theorem proved above, together with the lemmas proved in the first section, show that for odd initial data the (destabilized) Kuramoto-Sivashinsky equation remains bounded in L_2 for all time. These results can be extended to arbitrary mean-zero initial data in the manner first done by Collet, Eckmann, Epstein and Stubbe[1] or by Goodman[10]. This paper was written in such a way as to be compatible with the results of Collet et. al, where one allows the potential ϕ_x to translate via a gradient-flow type dynamics. In particular Theorem 1 in this paper is stated in such as way as to be compatible with Lemma 5.1 in [1]. From this the results of Proposition 4.3 in [1] follow, and the L_2 boundedness result extends to arbitrary mean-zero initial data. One could equally well employ the related idea of Goodman, and look at the rate of change of the distance of u from the set of all translates of ϕ.

3 CONCLUSIONS

In this paper we have constructed a function ϕ such that the ball $B(\phi, cL^{\frac{3}{2}})$ is a global attracting set for the (destabilized) Kuramoto-Sivashinsky equation

$$u_t = -u_{xxxx} - u_{xx} + \gamma u + uu_x.$$

This result has the best possible scaling with L, the size of the domain, since the above equation has stationary solutions with L_2 norm which scales like $L^{\frac{3}{2}}$.

While the result of Giacomelli and Otto is slightly stronger, the Lyapunov function argument outlined here is still interesting for a number of reasons. First is that it gives the optimal scaling for L_2 boundedness of the destabilized KS equation, a result that seems to be new. Secondly, this calculation shows that the critical exponents for an argument of this type can be achieved, and is interesting as a demonstration of the limits of this kind of Lyapunov function argument. Finally, it should be noted that there is an elegant result of Molinet[17], improving on earlier work of Sell and Taboada[19], which gives L_2 boundedness of the Kuramoto-Sivashinsky equation in two spatial dimensions

$$\vec{u}_t = -\Delta^2 \vec{u} - \Delta \vec{u} + \vec{\nabla}(\vec{u} \cdot \vec{u}) \qquad \vec{\nabla}^{\perp} \cdot \vec{u} = 0$$

for sufficiently thin rectangular domains. The results of Molinet require the construction of a Lyapunov function for the problem one spatial dimension. In the original paper Molinet uses the Lyapunov function constructed by Collet et. al. to show boundedness in $L_2[(0, L_x) \times (0, L_y)]$ assuming that the width in the second spatial direction satisfies (for $L_x >> 1, L_y << 1$)

$$L_y \leq C L_x^{-\frac{67}{35}}.$$

Assuming the above condition holds, together with a bound on the L_2 of the initial data, Molinet establishes a long-time bound on the L_2 norm of the form

$$\limsup_{t \to \infty} ||\vec{u}||_2 \leq C L_x^{\frac{8}{5}} L_y^{\frac{1}{2}}.$$

In fact Molinet shows a great deal more, including decay in time of the second component of $\vec{u}$ and explicit estimates of the relevant constants. Using the Lyapunov function constructed here and applying Molinet's results verbatim gives L_2 boundedness assuming the aspect ratio satisfies

$$L_y \leq C L_x^{-\frac{13}{7}}.$$

This, in turn, leads to a bound on the L_2 norm of the form

$$\limsup_{t \to \infty} ||\vec{u}||_2 \leq C L_x^{\frac{3}{2}} L_y^{\frac{1}{2}}.$$

Acknowledgements: The authors would like to thank Dirk Hundertmark, Ralf Wittenberg and Jonathan Goodman for many useful conversations. In particular we would like to thank Ralf Wittenberg for pointing out to us the significance of the destabilized Kuramoto-Sivashinsky equation, and the connection to the background flow method. This research was supported by NSF grants DMS-0203938 and DMS-0354373. JCB would like to thank the Stanford University mathematics department for hospitality during the writing of part of this paper.

References

[1] Collet, P.,Eckmann, J.-P.,Epstein, H. and Stubbe, J. (1993). A Global Attracting Set for the Kuramoto-Sivashinsky Equation. *Comm. Math. Phys.* **152**, 203-214.

[2] Constantin, P. and Doering, C. (1992). Energy dissipation in shear driven turbulence. *Phys. Rev. Lett.* **69** (11), 1648-1651.

[3] Constantin, P. and Doering, C. (1995). Variational Bounds for Dissipative Systems. *Phys. D*, **82** (3), 221-228.

[4] Cox, S. and Overton, M. (1992). On the optimal design of columns against buckling. *SIAM J. Math. Anal.* **23** (2), 287-325.

[5] Foias, C.,Manley, O. and Temam, R. (1987). Attractors for the Bénard problem: Existence and physical bounds on their fractal dimensions. *Nonlinear Anal.* **11**, 939-967.

[6] Foias, C., Sell, G. R., and Temam, R. (1988). Inertial manifolds for nonlinear evolutionary equations. *J. Diff. Eq.* **73** (2), 309-353.

[7] Foias, C., Nicolaenko, B., Sell, G.R., and Temam, R. (1988). Inertial Manifolds for the Kuramoto-Sivashinsky equation and an estimate of their lowest dimension. *J. Math. Pures. Appl.* **67** (3), 197-226.

[8] Gambill, T. Ph.D. Thesis (In Preparation).

[9] Giacomelli, L., and Otto, F. (2005). New bounds for the Kuramoto-Sivashinsky equation. *Comm. Pure Appl. Math.* **58** (3), 297-318.

[10] Goodman, J. (1994). Stability of the Kuramoto-Sivashinsky and Related Systems. *Comm. Pure Appl. Math.* **47** (3), 293-306.

[11] Il´yashenko, Yu. S. (1992). Global analysis of the phase portrait for the Kuramoto-Sivashinsky equation. *J. Dynam. Diff. Eq.* **4** (4), 585-615.

[12] Johnson, M. E., Jolly, M. S., and Kevrekidis, I. G. (2001). The Oseberg transition: visualization of global bifurcations for the Kuramoto-Sivashinsky equation. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **11** (1), 1-18.

[13] Kevrekidis, I. G.; Nicolaenko, B.; Scovel, J. C. (1990). Back in the saddle again: a computer assisted study of the Kuramoto-Sivashinsky equation. *SIAM J. Appl. Math.* **50**, (3) 760-790.

[14] Kuramoto, Y., and Tsuzuki, T. (1976). Persistent propagation of concentration waves in dissipative media far from thermal equilibrium. *Prog. Theor. Phys.* **55** (2), 356-369.

[15] LaQuey, R., Mahajan, S., Rutherford, P. and Tang, W. (1975). *Phys. Rev. Lett* **34** (7) 391-394.

[16] Manneville, P. (1989). *Dissipative Structures and Weak Turbulence* Academic Press, San Francisco/London.

[17] Molinet,L. (2000). Local Dissipativity in L^2 for the Kuramoto-Sivashinsky Equation in Spatial Dimension 2, *J. Dynam. Diff. Eq.* **12** (3), 533-555.

[18] Nicolaenko B.,Scheurer, B., and Temam, R. (1985). Some Global Dynamical Properties of the Kuramoto-Sivashinsky Equations: Nonlinear Stability and Attractors. *Phys. D* **16** (2), 155-183.

[19] Sell, G. and Taboada, M. (1992). Local dissipativity and attractors for the Kuramoto-Sivashinsky equation in thin 2D domains. *Nonlinear Analysis* **18** (7), 671-687.

[20] Sivashinsky, G. (1980). On the flame propagation under conditions of stochiometry. *SIAM J. Appl. Math* **75**, 67-82.

[21] Temam, Roger (1988) *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer, Berlin/Heidelberg/New York.

[22] Tadjbakhsh, I. and Keller, J. B. (1962) Strongest columns and isoperimetric inequalities for eigenvalues. *Trans. ASME Ser. E. J. Appl. Mech.* **29**, 159-164.

[23] Wittenberg, R. W.(2002). Dissipativity, analyticity and viscous shocks in the (de)stabilized Kuramoto-Sivashinsky equation. *Phys. Lett. A* **300** (4-5), 407-416.