Chebyshev - Fourier expansion

- Chebyshev expansion is used in the radial direction and a Fourier expansion is used in the azimuthal direction

$$\psi(r,\theta) = \sum_{n=0}^{N}\sum_{m=0}^{M}\psi_{nm}T_m(r)\mathrm{e}^{in\theta}$$

Use of convolution to solve nonlinear terms

- The FFT provides a fast method of solving nonlinear terms in point space within the numerical solver
- The Eigensolver uses convolution for nonlinear terms

Homogeneous system

$$\tfrac{1}{r}(\partial_r\omega\partial_\theta\psi - \partial_r\psi\partial_\theta\omega) = \tfrac{1}{Re}\left(\partial_r^2\omega + \tfrac{1}{r}\partial_r\omega + \tfrac{1}{r^2}\partial_\theta^2\omega + \tfrac{8}{h^2}(\tilde{\omega}^* - \omega)\right)$$

$$\partial_r^2\psi + \tfrac{1}{r}\partial_r\psi + \tfrac{1}{r^2}\partial_\theta^2\psi = -\omega$$

Perturbation expansions

$\psi = \psi_0(r) + \varepsilon\psi_1(r)\mathrm{e}^{\lambda t+in\theta}\omega = \omega_0(r) + \varepsilon\omega_1(r)\mathrm{e}^{\lambda t+in\theta}$ – Linear expansions

- Independent of the Fourier mode

Resulting Equations

$$\mathbf{R}^2\omega_1\lambda + in\mathbf{R}(\partial_r\omega_0\psi_1 - \partial_r\psi_0\omega_1) = \tfrac{1}{Re}(\mathbf{R}^2\mathbf{D}^2\omega_1 + \mathbf{R}\mathbf{D}\omega_1 - n^2\mathbf{I}\omega_1 - \tfrac{8}{h^2}\mathbf{R}^2\omega_1)$$
$$\mathbf{R}^2\mathbf{D}^2\psi_1 + \mathbf{R}\mathbf{D}\psi_1 - n^2\mathbf{I}\psi_1 = -\mathbf{R}^2\omega_1$$

Eigenvalue problem

$$\begin{bmatrix} \tfrac{1}{Re}(\mathbf{R}^2\mathbf{D}^2 + \mathbf{R}\mathbf{D} - n^2\mathbf{I} - \tfrac{8}{h^2}\mathbf{R}^2) + in\mathbf{R}\overline{\overline{\partial_r\psi_0}} & -in\mathbf{R}\overline{\overline{\partial_r\omega_0}} \\ \mathbf{R}^2 & \mathbf{R}^2\mathbf{D}^2 + \mathbf{R}\mathbf{D} - n^2\mathbf{I} \end{bmatrix}\xi = \lambda\begin{bmatrix} \mathbf{R}^2 & 0 \\ 0 & 0 \end{bmatrix}\xi$$

Reynold's number definition for the equations (as above)

$Re = \frac{a\Delta\Omega}{v}$ Note: the shear layer thickness, h, is not in this definition. I inaccurately write this into the equations as $\frac{1}{Re}$, but I don't want to put the h into the other terms. Maybe I should call it mRe (mostly the Reynold's number).

Reynold's number definition for the instabilities (this is the correct definition)

$Re = \frac{ah\Delta\Omega}{v}$ a - distance to shear layer (.5), h - shear layer thickness (.05)

$\Delta\Omega$ – angular velocity, v - viscosity of the fluid (.002)

Instabilities of the homogeneous problem

Mode 1: have not found this one
Mode 2: 70.0
Mode 3: 46.5
Mode 4: 41.2
Mode 5: 38.4
Mode 6: 37.8
Mode 7: 38.8
Mode 8: 41.3
Mode 9: 45.3
Mode 10: 51.8

Nonhomogeneous system

$$\lambda\omega_1 e^{\lambda t} + \frac{1}{r}\left(\partial_r\omega_1\partial_\theta\psi_0 e^{\lambda t} - \partial_r\psi_0\partial_\theta\omega_1 e^{\lambda t} + \partial_r\omega_0\partial_\theta\psi_1 e^{\lambda t} - \partial_r\psi_1\partial_\theta\omega_0 e^{\lambda t}\right) = \frac{1}{Re}\left[\Delta\omega_1 e^{\lambda t} - \frac{8}{h^2}\omega_1 e^{\lambda t}\right]$$

$$\Delta\psi_1 e^{\lambda t} = -\omega_1 e^{\lambda t}$$

Perturbation expansions

$$\psi(r,\theta,t) = \psi_0(r,\theta) + \varepsilon\psi_1(r,\theta)e^{\lambda t}$$

$$\omega(r,\theta,t) = \omega_0(r,\theta) + \varepsilon\omega_1(r,\theta)e^{\lambda t}$$

- Integrated Chebyshev/Fourier expansion

$$\zeta(r,\theta) = \begin{bmatrix} \zeta_{\cos 0}(r) \\ \zeta_{\cos 1}(r) \\ \vdots \\ \zeta_{\cos N}(r) \\ 0 \\ \zeta_{\sin 1}(r) \\ \zeta_{\sin 2}(r) \\ \vdots \\ \zeta_{\sin N}(r) \end{bmatrix}$$ each $\zeta_{\cos j}(r)/\zeta_{\sin j}(r)$ has a Chebyshev modal expansion

Resulting equations

$$
\begin{aligned}
&\lambda \hat{\mathbf{R}}^2 \omega_1 + \hat{\mathbf{R}}\hat{\mathbf{Q}}\psi_0 \hat{\mathbf{D}}\omega_1 - \hat{\mathbf{R}}\hat{\mathbf{D}}\psi_0 \hat{\mathbf{Q}}\omega_1 + \hat{\mathbf{R}}\hat{\mathbf{D}}\omega_0 \hat{\mathbf{Q}}\psi_1 - \hat{\mathbf{R}}\hat{\mathbf{Q}}\omega_0 \hat{\mathbf{D}}\psi_1 = \\
&\quad \tfrac{1}{Re}\Big[\hat{\mathbf{R}}^2\hat{\mathbf{D}}^2\omega_1 + \hat{\mathbf{R}}\hat{\mathbf{D}}\omega_1 + \hat{\mathbf{Q}}^2\omega_1 - \hat{\mathbf{R}}^2\tfrac{8}{h^2}\omega_1\Big] \\
&\hat{\mathbf{R}}^2\hat{\mathbf{D}}^2\psi_1 + \hat{\mathbf{R}}\hat{\mathbf{D}}\psi_1 + \hat{\mathbf{Q}}^2\psi_1 = -\hat{\mathbf{R}}^2\omega
\end{aligned}
$$

Full eigenvalue problem

$$
\begin{bmatrix}
\frac{1}{Re}\left(\hat{\mathbf{R}}^2\hat{\mathbf{D}}^2 + \hat{\mathbf{R}}\hat{\mathbf{D}} + \hat{\mathbf{Q}}^2 - \hat{\mathbf{R}}^2\frac{8}{h^2}\right) - \hat{\mathbf{R}}\hat{\mathbf{Q}}\psi_0\hat{\mathbf{D}} + \hat{\mathbf{R}}\hat{\mathbf{D}}\psi_0\hat{\mathbf{Q}} & \hat{\mathbf{R}}\hat{\mathbf{Q}}\omega_0\hat{\mathbf{D}} - \hat{\mathbf{R}}\hat{\mathbf{D}}\omega_0\hat{\mathbf{Q}} \\
\hat{\mathbf{R}}^2 & \hat{\mathbf{R}}^2\hat{\mathbf{D}}^2 + \hat{\mathbf{R}}\hat{\mathbf{D}} + \hat{\mathbf{Q}}^2
\end{bmatrix}\xi =
\lambda\begin{bmatrix}
\hat{\mathbf{R}}^2 & 0 \\
0 & 0
\end{bmatrix}\xi
$$