Intersection of Ellipses

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1 Introduction

This article describes how to compute the points of intersection of two ellipses, a geometric query labeled *find intersections*. It also shows how to determine if two ellipses intersect without computing the points of intersection, a geometric query labeled *test intersection*. Specifically, the geometric queries for the ellipses E_0 and E_1 are:

- *Find Intersections.* If E_0 and E_1 intersect, find the points of intersection.
- *Test Intersection.* Determine if
  - E_0 and E_1 are separated (there exists a line for which the ellipses are on opposite sides),
  - E_0 properly contains E_1 or E_1 properly contains E_0, or
  - E_0 and E_1 intersect.

An implementation of the *find* query, in the event of no intersections, might not necessarily determine if one ellipse is contained in the other or if the two ellipses are separated. Let the ellipses E_i be defined by the quadratic equations

$$\begin{aligned} Q_i(\vec{X}) &= \vec{X}^T A_i \vec{X} + \vec{B}_i^T \vec{X} + C_i \\ &= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a_{00}^{(i)} & a_{01}^{(i)} \\ a_{01}^{(i)} & a_{11}^{(i)} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_0^{(i)} & b_1^{(i)} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + c^{(i)} \\ &= 0 \end{aligned}$$

for $i = 0, 1$. It is assumed that the A_i are positive definite. In this case, $Q_i(\vec{X}) < 0$ defines the inside of the ellipse and $Q_i(\vec{X}) > 0$ defines the outside.

2 Find Intersection

The two polynomials $f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ and $g(x) = \beta_0 + \beta_1 x + \beta_2 x^2$ have a common root if and only if the Bézout determinant is zero,

$$(\alpha_2\beta_1 - \alpha_1\beta_2)(\alpha_1\beta_0 - \alpha_0\beta_1) - (\alpha_2\beta_0 - \alpha_0\beta_2)^2 = 0.$$

This is constructed by the combinations

$$0 = \alpha_2 g(x) - \beta_2 f(x) = (\alpha_2\beta_1 - \alpha_1\beta_2)x + (\alpha_2\beta_0 - \alpha_0\beta_2)$$

and

$$0 = \beta_1 f(x) - \alpha_1 g(x) = (\alpha_2\beta_1 - \alpha_1\beta_2)x^2 + (\alpha_0\beta_1 - \alpha_1\beta_0),$$

solving the first equation for x and substituting it into the second equation. When the Bézout determinant is zero, the common root of $f(x)$ and $g(x)$ is

$$\bar{x} = \frac{\alpha_2\beta_0 - \alpha_0\beta_2}{\alpha_1\beta_2 - \alpha_2\beta_1}.$$

The common root to $f(x) = 0$ and $g(x) = 0$ is obtained from the linear equation $\alpha_2 g(x) - \beta_2 f(x) = 0$ by solving for x.

The ellipse equations can be written as quadratics in x whose coefficients are polynomials in y,

$$Q_i(x, y) = \left(a_{11}^{(i)} y^2 + b_1^{(i)} y + c^{(i)}\right) + \left(2a_{01}^{(i)} y + b_0^{(i)}\right) x + \left(a_{00}^{(i)}\right) x^2.$$

Using the notation of the previous paragraph with f corresponding to Q_0 and g corresponding to Q_1,

$$\alpha_0 = a_{11}^{(0)} y^2 + b_1^{(0)} y + c^{(0)}, \quad \alpha_1 = 2a_{01}^{(0)} y + b_0^{(0)}, \quad \alpha_2 = a_{00}^{(0)},$$
$$\beta_0 = a_{11}^{(1)} y^2 + b_1^{(1)} y + c^{(1)}, \quad \beta_1 = 2a_{01}^{(1)} y + b_0^{(1)}, \quad \beta_2 = a_{00}^{(1)}.$$

The Bézout determinant is a quartic polynomial $R(y) = u_0 + u_1 y + u_2 y^2 + u_3 y^3 + u_4 y^4$ where

$$\begin{aligned}
u_0 &= v_2 v_{10} - v_4^2 \\
u_1 &= v_0 v_{10} + v_2(v_7 + v_9) - 2v_3 v_4 \\
u_2 &= v_0(v_7 + v_9) + v_2(v_6 - v_8) - v_3^2 - 2v_1 v_4 \\
u_3 &= v_0(v_6 - v_8) + v_2 v_5 - 2v_1 v_3 \\
u_4 &= v_0 v_5 - v_1^2
\end{aligned}$$

with

$$\begin{aligned}
v_0 &= 2\left(a_{00}^{(0)} a_{01}^{(1)} - a_{00}^{(1)} a_{01}^{(0)}\right) \\
v_1 &= a_{00}^{(0)} a_{11}^{(1)} - a_{00}^{(1)} a_{11}^{(0)} \\
v_2 &= a_{00}^{(0)} b_0^{(1)} - a_{00}^{(1)} b_0^{(0)} \\
v_3 &= a_{00}^{(0)} b_1^{(1)} - a_{00}^{(1)} b_1^{(0)} \\
v_4 &= a_{00}^{(0)} c^{(1)} - a_{00}^{(1)} c^{(0)} \\
v_5 &= 2\left(a_{01}^{(0)} a_{11}^{(1)} - a_{01}^{(1)} a_{11}^{(0)}\right) \\
v_6 &= 2\left(a_{01}^{(0)} b_1^{(1)} - a_{01}^{(1)} b_1^{(0)}\right) \\
v_7 &= 2\left(a_{01}^{(0)} c^{(1)} - a_{01}^{(1)} c^{(0)}\right) \\
v_8 &= a_{11}^{(0)} b_0^{(1)} - a_{11}^{(1)} b_0^{(0)} \\
v_9 &= b_0^{(0)} b_1^{(1)} - b_0^{(1)} b_1^{(0)} \\
v_{10} &= b_0^{(0)} c^{(1)} - b_0^{(1)} c^{(0)}
\end{aligned}$$

For each $\bar{y}$ solving $R(\bar{y}) = 0$ solve $Q_0(x, \bar{y}) = 0$ for up to two values $\bar{x}$. Eliminate any *false solution* $(\bar{x}, \bar{y})$ by verifying that $P_i(\bar{x}, \bar{y}) = 0$ for $i = 0, 1$.

3 Test Intersection

3.1 Variation 1

All level curves defined by $Q_0(x, y) = \lambda$ are ellipses, except for the minimum (negative) value λ for which the equation defines a single point, the center of every level curve ellipse. The ellipse defined by $Q_1(x, y) = 0$ is a curve that generally intersects many level curves of Q_0. The problem is to find the minimum level value λ_0 and maximum level value λ_1 attained by any (x, y) on the ellipse E_1. If $\lambda_1 < 0$, then E_1 is properly contained in E_0. If $\lambda_0 > 0$, then E_0 and E_1 are separated. Otherwise, $0 \in [\lambda_0, \lambda_1]$ and the two ellipses intersect.

This can be formulated as a constrained minimization that can be solved by the method of Lagrange multipliers: Minimize $Q_0(\vec{X})$ subject to the constraint $Q_1(\vec{X}) = 0$. Define $F(\vec{X}, t) = Q_0(\vec{X}) + tQ_1(\vec{X})$. Differentiating yields $\vec{\nabla} F = \vec{\nabla} Q_0 + t\vec{\nabla} Q_1$ where the gradient indicates the derivatives in $\vec{X}$. Also, $\partial F/\partial t = Q_1$. Setting the t–derivative equal to zero reproduces the constraint $Q_1 =$. Setting the $\vec{X}$–derivative equal to zero yields $\vec{\nabla} Q_0 + t\vec{\nabla} Q_1 = \vec{0}$ for some t. Geometrically this means that the gradients are parallel.

Note that $\vec{\nabla} Q_i = 2A_i\vec{X} + \vec{B}_i$, so

$$\vec{0} = \vec{\nabla} Q_0 + t\vec{\nabla} Q_1 = 2(A_0 + tA_1)\vec{X} + (\vec{B}_0 + t\vec{B}_1).$$

Formally solving for $\vec{X}$ yields

$$\vec{X} = -(A_0 + tA_1)^{-1}(\vec{B}_0 + t\vec{B}_1)/2 = \frac{1}{\delta(t)}\vec{Y}(t)$$

where $\delta(t)$ is the determinant of $(A_0 + tA_1)$, a quadratic polynomial in t, and $\vec{Y}(t)$ has components quadratic in t. Replacing this in $Q_1(\vec{X}) = 0$ yields

$$\vec{Y}(t)^T A_1 \vec{Y}(t) + \delta(t)\vec{B}_1^T \vec{Y}(t) + \delta(t)^2 C_1 = 0,$$

a quartic polynomial in t. The roots can be computed, the corresponding values of $\vec{X}$ computed, and $Q_0(\vec{X})$ evaluated. The minimum and maximum values are stored as λ_0 and λ_1, and the earlier comparisons with zero are applied.

This method leads to a quartic polynomial, just as the *find* query did. But this query does answer questions about the relative positions of the ellipses (separated or proper containment) when the *find* query indicates that there is no intersection.

3.2 Variation 2

A less expensive *test* query is based on the *find* query, but cannot answer the question of proper containment or separation when there is no intersection. Rather than solve the quartic equation $R(y) = 0$ that was derived in the section on finding intersections, it is enough to determine if $R(y)$ has any real roots. The ellipses intersect if and only if there are real roots. If $u_4 = 0$ and $u_3 = 0$, then there are real roots as long as $u_1^2 - 4u_0u_2 \geq 0$. If $u_4 = 0$ and $u_3 \neq 0$, then the cubic polynomial necessarily has a real root. If $u_4 \neq 0$, then multiply the equation, if necessary, by -1 to make the leading coefficient positive. The polynomial has no real roots if and only if $R(y) > 0$ for all y. It is enough to compute the local minima of R and show they are all positive. This requires finding the roots of the cubic polynomial $R'(y) = 0$ and evaluating $R(y)$ and testing if it is positive at those roots.

But it is even possible to avoid finding roots whatsoever. This uses the method of bounding roots by Sturm sequences. Consider a polynomial $f(t)$ defined on interval $[a, b]$. A Sturm sequence for f is a set of polynomials $f_i(t)$, $0 \leq i \leq m$ such that Degree$(f_{i+1}) >$ Degree(f_i) and the number of distinct real roots for f in $[a, b]$ is $N = s(a) - s(b)$ where $s(a)$ is the number of sign changes of $f_0(a), \ldots, f_m(a)$ and $s(b)$ is the number of sign changes of $f_1(b), \ldots, f_m(b)$. The total number of real–valued roots of f on $\mathbb{R}$ is $s(-\infty) - s(\infty)$. It is not always the case that $m =$ Degree(f). The classic Sturm sequence is $f_0(t) = f(t)$, $f_1(t) = f'(t)$, and $f_i(t) = -$Remainder(f_{i-2}/f_{i-1}) for $i \geq 2$. The polynomials are generated by this method until the remainder term is a constant. This method is applied to $R(y)$ on $(-\infty, \infty)$ to determine the number of real roots.

3.3 Variation 3

This test is similar to variation 2, but it requires that one of the ellipses be axis–aligned (let it be E_0 for the argument). It is possible to force this to happen by an affine change of variables, the correct transformation requiring determining the eigenvalues of A_0, an operation that involves solving a quadratic equation. If the application already knows the axes of the ellipses, then this only reduces the computation time. I believe this argument also shows that $R(y)$ can never be cubic, only quadratic or quartic.

The quadratic equation for the axis–aligned ellipse can be written as $(y - y_0)^2 = a_0 + a_1x + a_2x^2$ where $a_2 < 0$. The other ellipse equation can be written as $(y - y_0)^2 + (b_{10} + b_{11}x)(y - y_0) + (b_{00} + b_{01}x + b_{02}x^2) = 0$. Substituting $(y - y_0)^2$ from the first equation into the second one, solving the second for $(y - y_0)$, replacing it in the first, and cross–multiplying leads to the polynomial $P(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$ where

$$
\begin{aligned}
c_0 &= (b_{00} + a_0)^2 - a_0b_{10}^2 \\
c_1 &= 2(b_{00} + a_0)(b_{01} + a_1) - 2a_0b_{10}b_{11} - a_1b_{10}^2 \\
c_2 &= 2(b_{00} + a_0)(b_{02} + a_2) + (b_{01} + a_1)^2 - a_0b_{11}^2 - a_2b_{10}^2 - 2a_1b_{10}b_{11} \\
c_3 &= 2(b_{01} + a_1)(b_{02} + a_2) - a_1b_{11}^2 - 2a_2b_{10}b_{11} \\
c_4 &= (b_{02} + a_2)^2 - a_2b_{11}^2
\end{aligned}
$$

Since $a_2 < 0$, the only way $c_4 = 0$ is if $a_2 = -b_{02}$ and $b_{11} = 0$. In this case, $c_3 = 0$ is forced. If both $c_4 = c_3 = 0$, then $c_2 = (b_{01} + a_1)^2 - a_2b_{10}^2$. The only way $c_2 = 0$ is if $a_1 = -b_{01}$ and $b_{10} = 0$. In this case, $c_1 = 0$ is forced and the polynomial is $c_0 = 0$, finally leading to $P(x)$ being identically zero. The two quadratic equations are for the same ellipse. So the only three cases to trap in the code are via the Boolean short circuit, $c_4 \neq 0$ or $c_2 \neq 0$ or ellipses are the same. The hard case is $c_4 \neq 0$, but as in variation 2, it is enough just to argue whether or not $P(y)$ has roots. This only requires solving a cubic polynomial equation $P'(y) = 0$ and testing the values of $P(y)$.

The method of Sturm sequences, as shown in variation 2, can also be applied here for the fastest possible *test* query.