Higher Order Flexibility of Octahedra

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Abstract

More than hundred years ago R. Bricard determined all continuously flexible octahedra. On the other hand, also the geometric characterization of first-order flexible octahedra has been well known for a long time. The objective of this paper is to analyze the cases between, i.e., octahedra which are infinitesimally flexible of order $n > 1$ but not continuously flexible. We prove explicit necessary and sufficient conditions for the orders two, three and even for all $n < 8$, provided the octahedron under consideration is not totally flat. Any order ≥ 8 implies already continuous flexibility, as the configuration problem for octahedra is of degree eight.

Key Words: Infinitesimal flexibility, flexible polyhedra, octahedra
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1 Introduction

Let $\mathbf{F}$ be a framework in the Euclidean d-space $\mathbb{E}^d$ with *vertex set* V and *edge set* E, i.e.,

$$V = \{\mathbf{p}_1, \ldots, \mathbf{p}_v\},\ \mathbf{p}_i \in \mathbb{R}^d \quad \forall i \in I := \{1, \ldots, v\} \quad \text{and} \quad E \subset \Big\{(i,j) \mid i < j,\ (i,j) \in I^2\Big\}.$$

For each edge $\mathbf{p}_i\mathbf{p}_j$ of $\mathbf{F}$ the Euclidean length l_{ij} obeys

$$(1) \qquad f_{ij}(\mathbf{p}_i, \mathbf{p}_j) := \|\mathbf{p}_i - \mathbf{p}_j\| - l_{ij} = 0 \quad \forall (i,j) \in E.$$

We suppose $l_{ij} > 0$ for all edges of $\mathbf{F}$.

Definition: The framework $\mathbf{F} = (V, E)$ is *infinitesimally flexible of order* n (in the classical sense) if and only if for each $k \in I$ there is a polynomial function

$$(2) \qquad \mathbf{x}_k(t) := \mathbf{p}_k + t\,\mathbf{x}_{k,1} + \ldots + t^n \mathbf{x}_{k,n}\,, \quad n \geq 1,$$

such that

- the replacement of $\mathbf{p}_i$ and $\mathbf{p}_j$ in (1) by $\mathbf{x}_i(t)$ and $\mathbf{x}_j(t)$, resp., gives functions obeying

  $$(3) \qquad f_{ij}\left(\mathbf{x}_i(t), \mathbf{x}_j(t)\right) = \|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| - l_{ij} = o(t^n) \quad \forall (i,j) \in E,$$

  i.e., with a zero of multiplicity $\geq n{+}1$ at $t = 0$, and

- in order to exclude *trivial* flexes, the vectors $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{v,1}$ are not the velocity vectors of the vertices $\mathbf{p}_1, \ldots, \mathbf{p}_v$, resp., under any motion of $\mathbf{F}$ as a rigid body.

The v-tupel of functions $(\mathbf{x}_1(t), \ldots, \mathbf{x}_v(t))$ is called a nontrivial *n-th-order flex* of $\mathbf{F}$.[1]

Suppose, the framework $\mathbf{F}$ is given by its combinatorial structure E and by the lengths l_{ij} of its edges. Then (1) defines a system of $e := \#E$ quadratic equations

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) = l_{ij}^2 \quad \forall (i,j) \in E \tag{4}$$

for the $d \cdot v$ unknown coordinates of the vertices. After substituting (2) in (4) the comparison of coefficients of $t, t^2, \ldots, t^n$ gives rise to the following systems of linear equations, each for all $(i,j) \in E$, for the unknown $\mathbf{x}_{k,r} \in \mathbb{R}^d$, $k = 1, \ldots, v$ and $r \in \{1, \ldots, n\}$:

$$\begin{aligned}
&(5)\quad (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{x}_{i,1} - \mathbf{x}_{j,1}) &=&\ 0\,,\\
&(6)\quad (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{x}_{i,2} - \mathbf{x}_{j,2}) &=&\ -\tfrac{1}{2}(\mathbf{x}_{i,1} - \mathbf{x}_{j,1}) \cdot (\mathbf{x}_{i,1} - \mathbf{x}_{j,1}),\\
&(7)\quad (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{x}_{i,3} - \mathbf{x}_{j,3}) &=&\ -(\mathbf{x}_{i,1} - \mathbf{x}_{j,1}) \cdot (\mathbf{x}_{i,2} - \mathbf{x}_{j,2}),\\
&(8)\quad (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{x}_{i,4} - \mathbf{x}_{j,4}) &=&\ -(\mathbf{x}_{i,1} - \mathbf{x}_{j,1}) \cdot (\mathbf{x}_{i,3} - \mathbf{x}_{j,3}) - \tfrac{1}{2}(\mathbf{x}_{i,2} - \mathbf{x}_{j,2}) \cdot (\mathbf{x}_{i,2} - \mathbf{x}_{j,2}),
\end{aligned}$$

and so on. The $e \times vd$ matrix M of coefficients on the left side of these systems is always the same. It is called *rigidity matrix* of $\mathbf{F}$ (cf. [10]). If $(\tilde{\mathbf{x}}_{1,r}, \ldots, \tilde{\mathbf{x}}_{v,r})$ is any particular solution of the r-th linear system, $r \in \{1, \ldots, n\}$, then also

$$\mathbf{x}_{k,r} := \tilde{\mathbf{x}}_{k,r} + \mathbf{c} + C\mathbf{p}_k\,, \quad C^T = -C, \quad \text{for} \quad k = 1, \ldots, v \tag{9}$$

with constant $\mathbf{c} \in \mathbb{R}^d$ and any skew symmetric $d \times d$ matrix C solves this system.

In particular, the *"velocity vectors"* $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{v,1}$ of the vertices of $\mathbf{F}$ have to solve the homogeneous system (5). Therefore infinitesimal flexibility of order 1 can be characterized by the rank condition

$$\mathrm{rk}(M) < vd - \frac{d(d+1)}{2}\,. \tag{10}$$

Any nontrivial solution of (5) defines the right side in the inhomogeneous system (6), and then 2^{nd}-order flexibility of $\mathbf{F}$ is equivalent to the solvability of this system. This can be repeated step by step for (7), (8), ... to figure out the order of flexibility for any given framework.

2 Flexible Octahedra

In 1898 Raoul Bricard [4] proved that there are exactly three types of flexible octahedra[2] $\mathbf{O}$ in the Euclidean 3-space $\mathbb{E}^3$, the octahedra with line symmetry, those with planar symmetry and finally a particular third type which admits also two flat positions.

[1] In this "classical" definition the cases with trivial velocity vectors $\mathbf{x}_{1,1} = \cdots = \mathbf{x}_{v,1} = \mathbf{o}$ are excluded. This is a proper restriction since R. Connelly and H. Servatius [6] presented 1994 a framework which admits an analytical flex, but only under the assumption of vanishing initial velocities. Already a few years earlier I. Sabitov [9] proposed to denote the order of an infinitesimal flex by a pair (m, n) of numbers, the smallest and the highest exponent of t showing up in (2). Note also Remark 2 in [13].

[2] An octahedron is called *flexible* when its 1-skeleton can be flexibly embedded in $\mathbb{E}^3$. During the induced analytical self-motion edges are allowed to pass through one another.

Let $(\mathbf{p}_1, \mathbf{p}_2)$, $(\mathbf{q}_1, \mathbf{q}_2)$ and $(\mathbf{r}_1, \mathbf{r}_2)$ denote the pairs of opposite vertices of $\mathbf{O}$. We suppose that for all $i, j, k \in \{1, 2\}$ the vertices $\mathbf{p}_i\mathbf{q}_j\mathbf{r}_k$ are non-collinear. Then for the first of BRICARD's types all pairs of opposite vertices are symmetric with respect to a line. In the second case two pairs are symmetric with respect to a plane which passes through the two remaining vertices. The flat configurations of the third type have edges which are tangent to three concentric circles (see [5], Fig. 297) or which form three parallelograms (see [5], Fig. 298 or [12], Fig. 4). For further references see [12] where a new proof for the uniqueness of BRICARD's octahedra was given.

Actually there are two additional but *degenerate cases*[3]: One consists of a two-fold covered four-sided pyramid ($\mathbf{p}_1 = \mathbf{p}_2$), which trivially flexes like a spherical four-bar mechanism. At the fifth type two pairs of opposite vertices, e.g. $\mathbf{q}_1, \mathbf{q}_2, \mathbf{r}_1, \mathbf{r}_2$, are located on a line l. Hence this octahedron consists of two flat four-sided pyramides which can rotate independently about the common "basis line" l.

Both degenerate cases have also non-degenerate configurations which in general are rigid: In the first case this configuration consists of a double pyramid with a planar equator $\mathbf{q}_1\mathbf{r}_1\mathbf{q}_2\mathbf{r}_2$ and the remaining vertices $\mathbf{p}_1, \mathbf{p}_2$ being symmetric with respect to the equator plane (cf. [12], footnote 1 and case 1.2.1). The octahedra treated in [16] are of this type. In the second case the equator $\mathbf{q}_1\mathbf{r}_1\mathbf{q}_2\mathbf{r}_2$ is located on a one-sheet hyperboloid of revolution while $\mathbf{p}_1$ and $\mathbf{p}_2$ lie on the hyperboloid's axis (cf. [12], case 1.2.2). There are octahedra which at the same time belong to all three BRICARD types and whose flexions can also continuously blend into the trivial motions listed above as degenerate cases ([12], p. 53).

Octahedra made from cardboard can also look flexible when they are infinitesimally flexible or when they admit two isometric and sufficiently adjacent configurations. Suggestions for producing such 'flexible' models can be found in W. WUNDERLICH's paper [15]. Here he also proves by kinematic means a geometric characterization of octahedra with first-order infinitesimal flexibility (German: "Wackeloktaeder"). This is the equivalence "(i) $\Leftrightarrow$ (ii)" in the following

Theorem 1: *For an octahedron* $\mathbf{O}$ *with four non-coplanar* $\mathbf{q}_1, \mathbf{r}_1, \mathbf{q}_2, \mathbf{r}_2$ *and* $\mathbf{p}_1 \neq \mathbf{p}_2$ *any two of the following four statements are equivalent:*

(i) $\mathbf{O}$ *is infinitesimally flexible of first order.*

(ii) *There are two points* $\mathbf{s}_1, \mathbf{s}_2$ *such that* $\mathbf{p}_1\mathbf{q}_1\mathbf{r}_1\mathbf{s}_1$ *and* $\mathbf{p}_2\mathbf{q}_2\mathbf{r}_2\mathbf{s}_2$ *are two tetrahedra of* MÖBIUS *type which at the same time are mutually inscribed and circumscribed.*

(iii) *There are four pairwise edge-disjoint faces of* $\mathbf{O}$, *which span concurrent planes.*

(iv) *There is a second-order surface* Φ *passing through the vertices* $\mathbf{p}_1, \mathbf{p}_2$ *and through the sides of the skew quadrangle* $\mathbf{q}_1\mathbf{r}_1\mathbf{q}_2\mathbf{r}_2$.

The quadruples of edge-disjoint faces mentioned in (iii) are $\{\mathbf{p}_1\mathbf{q}_1\mathbf{r}_1,\ \mathbf{p}_2\mathbf{q}_2\mathbf{r}_1,\ \mathbf{p}_1\mathbf{q}_2\mathbf{r}_2,\ \mathbf{p}_2\mathbf{q}_1\mathbf{r}_2\}$ as well as $\{\mathbf{p}_2\mathbf{q}_2\mathbf{r}_2,\ \mathbf{p}_1\mathbf{q}_1\mathbf{r}_2,\ \mathbf{p}_2\mathbf{q}_1\mathbf{r}_1,\ \mathbf{p}_1\mathbf{q}_2\mathbf{r}_1\}$. The concurrence of the planes spanned by one of these quadruples implies already the concurrence of the other four. The common points are exactly $\mathbf{s}_1$ and $\mathbf{s}_2$ showing up in statement (ii).

[3]The author expresses his thanks to I. SABITOV for the valuable discussion held recently in this concern.

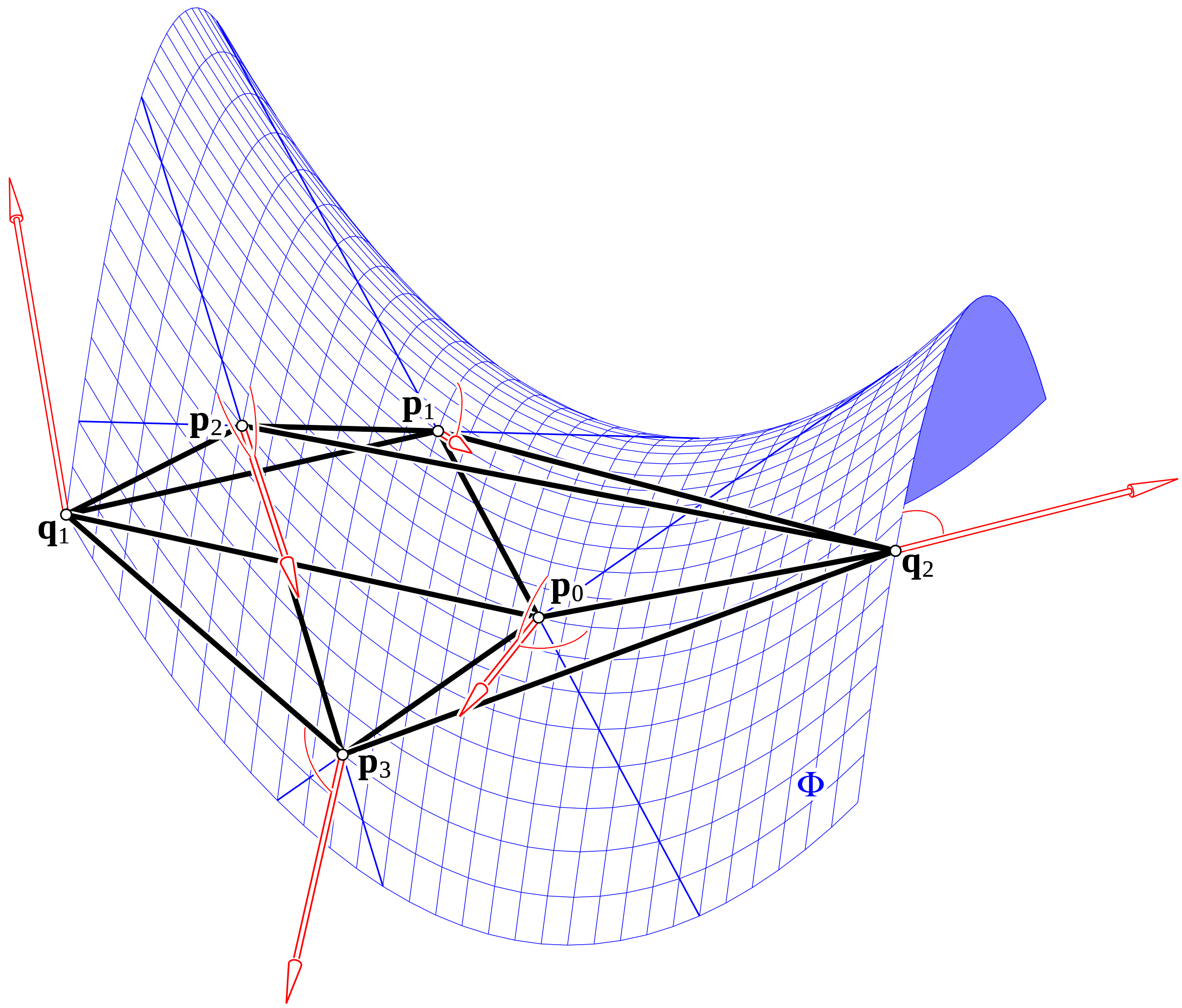

Figure 1: First-order flexible octahedron $\mathbf{O}$ with velocity vectors orthogonal to Φ

The statement (iv) (see Fig. 1) can be found in [1], p. 316, and in [12], p. 44. The equivalence "(i) $\Leftrightarrow$ (iv)" shows that *one can build a first-order flexible octahedron from any given five vertices* (compare Corollary 1 in Section 4). This is since the sides of the skew quadrangle $\mathbf{q}_1\mathbf{r}_1\mathbf{q}_2\mathbf{r}_2$ together with the vertex $\mathbf{p}_1$ define the second-order surface Φ of statement (iv) uniquely. The surface Φ is the geometric locus for the sixth vertex $\mathbf{p}_2$. For a planar quadrangle $\mathbf{q}_1\mathbf{r}_1\mathbf{q}_2\mathbf{r}_2$ it is sufficient to choose $\mathbf{p}_2$ in the affine span of this quadrangle. We will prove later with equation (23) that in the skew case the *velocity vectors* of the vertices can be specified *orthogonally* to the surface Φ.

3 Conditions for n-th-order flexibility

We change the notation used in the previous sections. We see the octahedron $\mathbf{O}$ as a *suspension* with the *equator* $\mathbf{p}_0, \ldots, \mathbf{p}_3$ and the two *poles* $\mathbf{q}_1, \mathbf{q}_2$. Now we subdivide the edge set E of $\mathbf{O}$ into the equator $\{\mathbf{p}_0\mathbf{p}_1, \ldots, \mathbf{p}_3\mathbf{p}_0\}$ and the set of edges $\mathbf{p}_i\mathbf{q}_j$, $i \in \{0, \ldots, 3\}$ and $j \in \{1, 2\}$. The latter form a bipartite sub-framework $\mathbf{O}'$ of $\mathbf{O}$. This is the reason why we can take over some of the arguments used in [14] for the analysis of a planar bipartite framework.

Let an n-th-order flex of $\mathbf{O}$ be given by

$$\mathbf{p}_i'(t) = \mathbf{p}_i + \mathbf{p}_{i,1}t + \cdots + \mathbf{p}_{i,n}t^n \quad \text{and} \quad \mathbf{q}_j'(t) = \mathbf{q}_j + \mathbf{q}_{j,1}t + \cdots + \mathbf{q}_{j,n}t^n \tag{11}$$

for $i = 0, \ldots, 3$ and $j = 1, 2$.

From now on we suppose *non-coplanar* $\{\mathbf{p}_0, \ldots, \mathbf{p}_3\}$: Then for each $t \in \mathbb{R}$ there is an affine transformation $\alpha : \mathbf{p}_i \mapsto \mathbf{p}_i'(t)$ for $i = 0, \ldots, 3$. When we use matrix notation and write the coordinate vectors as columns, then we can set up

$$\alpha(t)\colon \ \mathbf{p}_i \mapsto \mathbf{p}_i'(t) = \mathbf{a}(t) + A(t)\mathbf{p}_i \quad \text{with} \quad \mathbf{a}(0) = \mathbf{o}, \quad A(0) = I_3\,, \tag{12}$$

where I_3 denotes the 3×3 unit matrix. The coordinates of $\mathbf{a}(t)$ and the entries of $A(t)$ are polynomials of degree $\leq n$. In the new notation equation (3) reads

$$\left[\mathbf{p}_i'(t) - \mathbf{q}_j'(t)\right]^2 - (\mathbf{p}_i - \mathbf{q}_j)^2 = o(t^n) \quad \text{for all} \quad i \in \{0, \ldots, 3\} \quad \text{and} \quad j \in \{1, 2\}. \tag{13}$$

We subtract the equation for $i = 0$ and obtain

$$[\mathbf{p}_i'(t) - \mathbf{p}_0'(t)]^T \left[\mathbf{p}_i'(t) + \mathbf{p}_0'(t) - 2\mathbf{q}_j'(t)\right] - (\mathbf{p}_i - \mathbf{p}_0)^T(\mathbf{p}_i + \mathbf{p}_0 - 2\mathbf{q}_j) = o(t^n).$$

The difference of these equation for $j = 1, 2$ gives

$$-2\,[\mathbf{p}_i'(t) - \mathbf{p}_0'(t)]^T \,[\mathbf{q}_2'(t) - \mathbf{q}_1'(t)] + 2(\mathbf{p}_i - \mathbf{p}_0)^T(\mathbf{q}_2 - \mathbf{q}_1) = o(t^n),$$

and due to (12) we get

$$-2(\mathbf{p}_i - \mathbf{p}_0)^T A(t)^T \,[\mathbf{q}_2'(t) - \mathbf{q}_1'(t)] + 2(\mathbf{p}_i - \mathbf{p}_0)^T(\mathbf{q}_2 - \mathbf{q}_1) = o(t^n) \quad \text{for} \quad i \in \{1, 2, 3\}.$$

The supposed linear independence of the difference vectors $\{\mathbf{p}_1 - \mathbf{p}_0, \mathbf{p}_2 - \mathbf{p}_0, \mathbf{p}_3 - \mathbf{p}_0\}$ implies

$$A(t)^T\,[\mathbf{q}_2'(t) - \mathbf{q}_1'(t)] - (\mathbf{q}_2 - \mathbf{q}_1) = \mathbf{o}(t^n).$$

This means that for each $t \in \mathbb{R}$ there is a second affine transformation

$$\widehat{\alpha}(t)\colon \ \mathbf{q}_j'(t) \mapsto \widehat{\mathbf{a}}(t) + A(t)^T\mathbf{q}_j'(t) = \mathbf{q}_j + \mathbf{o}(t^n) \quad \text{for} \quad j = 1, 2 \tag{14}$$

with $\widehat{\mathbf{a}}(0) = \mathbf{o}$.

For t sufficiently near to 0 the image points $\{\mathbf{p}_0'(t), \ldots, \mathbf{p}_3'(t)\}$ are non-coplanar, too. Hence there exists the inverse

$$\widehat{\alpha}(t)^{-1}\colon \ \mathbf{q}_j \mapsto \mathbf{b}(t) + A(t)^{-1^T}\mathbf{q}_j = \mathbf{q}_j' + \mathbf{o}(t^n) \quad \text{with} \quad \mathbf{b} := -A^{-1^T}\widehat{\mathbf{a}} \tag{15}$$

We substitute in (13) the matrix representations (12) of α and (14) of $\widehat{\alpha}$ and obtain

$$\begin{aligned} &\mathbf{a}^T\mathbf{a} + 2\mathbf{a}^T(A\,\mathbf{p}_i - \mathbf{q}_j') \ + \ \mathbf{p}_i^T A^T A\,\mathbf{p}_i \ - \ 2\mathbf{p}_i^T A^T\mathbf{q}_j' + \mathbf{q}_j'^T\mathbf{q}_j' - \\ &-\widehat{\mathbf{a}}^T\widehat{\mathbf{a}} - 2\widehat{\mathbf{a}}^T(A^T\mathbf{q}_j' - \mathbf{p}_i) - \mathbf{q}_j'^T AA^T\mathbf{q}_j' + 2\mathbf{p}_i^T A^T\mathbf{q}_j' - \mathbf{p}_i^T\mathbf{p}_i = o(t^n). \end{aligned} \tag{16}$$

Separating the terms with $\mathbf{p}_i$ from those with $\mathbf{q}'_j(t)$ leads to

$$\begin{aligned}&\mathbf{p}_i^T(A^TA - I_3)\mathbf{p}_i + 2(\mathbf{a}^TA + \hat{\mathbf{a}}^T)\mathbf{p}_i + \mathbf{a}^T\mathbf{a} =\\ &= \mathbf{q}'^T_j(AA^T - I_3)\mathbf{q}'_j + 2(\mathbf{a}^T + \hat{\mathbf{a}}^TA^T)\mathbf{q}'_j + \hat{\mathbf{a}}^T\hat{\mathbf{a}} + o(t^n).\end{aligned}$$

As this equation must hold for all $i \in \{0, \dots, 3\}$ and $j \in \{1, 2\}$, both sides of the equation must be constant for each $t \in \mathbb{R}$. This results in two equations which are quadratic in the coordinates of $\mathbf{p}_i$ and $\mathbf{q}'_j$, respectively. With (15) we obtain – after subtracting $\mathbf{b}^T\mathbf{a} = \mathbf{a}^T\mathbf{b}$ from both sides –

$$\mathbf{p}_i^T(A^TA - I_3)\mathbf{p}_i + 2(\mathbf{a}^T - \mathbf{b}^T)A\,\mathbf{p}_i + (\mathbf{a}^T - \mathbf{b}^T)\mathbf{a} = \gamma(t), \quad i = 0, \dots, 3, \tag{17}$$

and for $j = 1, 2$

$$\mathbf{q}'^T_j(t)(AA^T - I_3)\mathbf{q}'_j(t) + 2(\mathbf{a}^T - \mathbf{b}^TAA^T)\mathbf{q}'_j(t) + \mathbf{b}^TAA^T\mathbf{b} - \mathbf{a}^T\mathbf{b} = \gamma(t) + o(t^n)$$

with a rational function $\gamma(t)$. For the sake of brevity we have ceased to indicate that the matrix A as well as the vectors $\mathbf{a}$ and $\mathbf{b}$ are functions of t.

Now we apply $\hat{\alpha}^{-1}$ to $\mathbf{q}'_j(t)$ in the second quadratic equation. Due to (15) we get finally

$$\mathbf{q}_j^T(I_3 - A^{-1}A^{-1^T})\mathbf{q}_j + 2(\mathbf{a}^T - \mathbf{b}^T)A^{-1^T}\mathbf{q}_j + (\mathbf{a}^T - \mathbf{b}^T)\mathbf{b} = \gamma(t) + o(t^n) \tag{18}$$

for $j = 1, 2$. Conversely, the two equations (17) and (18) imply (16) and hence (13).

Until now we have only dealt with the bipartite sub-framework $\mathbf{O}'$ of the octahedron. The edges $\mathbf{p}_i\mathbf{p}_{i+1}$ of the equator (subscripts modulo 4) impose the additional condition

$$\left[\mathbf{p}'_i(t) - \mathbf{p}'_{i+1}(t)\right]^2 - (\mathbf{p}_i - \mathbf{p}_{i+1})^2 = o(t^n) \quad \text{for} \quad i = 0, \dots, 3.$$

We substitute the representation (12) of α and obtain

$$(\mathbf{p}_i - \mathbf{p}_{i+1})^T(A^TA - I_3)(\mathbf{p}_i - \mathbf{p}_{i+1}) = o(t^n).$$

Due to (17) this is equivalent to

$$\mathbf{p}_i^T(A^TA - I_3)\mathbf{p}_{i+1} + (\mathbf{a}^T - \mathbf{b}^T)A(\mathbf{p}_i + \mathbf{p}_{i+1}) + (\mathbf{a}^T - \mathbf{b}^T)\mathbf{a} - \gamma(t) = o(t^n). \tag{19}$$

Due to (17) and (18) we define two bilinear functions

$$\begin{aligned}f(t; \mathbf{x}, \mathbf{y}) &:= \mathbf{x}^T(A^TA - I_3)\mathbf{y} + (\mathbf{a}^T - \mathbf{b}^T)A(\mathbf{x} + \mathbf{y}) + (\mathbf{a}^T - \mathbf{b}^T)\mathbf{a} - \gamma(t),\\ g(t; \mathbf{x}, \mathbf{y}) &:= \mathbf{x}^T(I_3 - A^{-1}A^{-1^T})\mathbf{y} + (\mathbf{a}^T - \mathbf{b}^T)A^{-1^T}(\mathbf{x} + \mathbf{y}) + (\mathbf{a}^T - \mathbf{b}^T)\mathbf{b} - \gamma(t).\end{aligned} \tag{20}$$

Then we can combine the equations (17), (18) and (19) in

Theorem 2: *The octahedron* $\mathbf{O}$ *with non-coplanar* $\{\mathbf{p}_0, \dots, \mathbf{p}_3\}$ *is flexible of order* n *if and only if in a neighborhood of* $t = 0$ *there are functions* $\mathbf{a}(t)$, $\mathbf{b}(t)$, $A(t)$, *and* $\gamma(t)$ *of class* C^n *such that the vertices* $\mathbf{p}_0, \dots, \mathbf{p}_3, \mathbf{q}_1, \mathbf{q}_2$ *obey the equations*

$$f(t; \mathbf{p}_i, \mathbf{p}_i) = 0, \quad f(t; \mathbf{p}_i, \mathbf{p}_{i+1}) = o(t^n), \quad g(t; \mathbf{q}_j, \mathbf{q}_j) = o(t^n)$$

for all $i \in \{0, \dots, 3\}$ *and* $j \in \{1, 2\}$ *with* f *and* g *according to* (20).

Similar to the proof of Lemma 2 in [14] we could show that for each t_0 sufficiently near to 0 the equations $f(t_0; \mathbf{x}, \mathbf{x}) = 0$ and $g(t_0; \mathbf{x}, \mathbf{x}) = 0$ represent two confocal surfaces of second order. The condition $f(t_0; \mathbf{p}_i, \mathbf{p}_{i+1}) = 0$ expresses conjugate position of two points $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ on the first of these two surfaces. This is equivalent to the fact that the connecting line is a generator of this surface.

Hence, if for any t_0 the equations $f(t_0; \mathbf{p}_i, \mathbf{p}_i) = f(t_0; \mathbf{p}_i, \mathbf{p}_{i+1}) = g(t_0; \mathbf{q}_j, \mathbf{q}_j) = 0$ hold for all $i = 0, \ldots, 3$ and $j = 1, 2$, then we have found a second-order surface passing through the sides of the skew quadrangle $\mathbf{p}_0, \ldots, \mathbf{p}_3$ such that there is a confocal surface passing through $\mathbf{q}_1$ and $\mathbf{q}_2$ (compare [11] or [12], p. 42, Satz 1). Whenever such a pair of surfaces is found, the theorem of IVORY gives a new octahedron $(\mathbf{p}_0(t_0), \ldots, \mathbf{q}_2(t_0))$ which is incongruent but isometric to the initial octahedron $(\mathbf{p}_0(0), \ldots, \mathbf{q}_2(0))$.

The conditions (17), (19) and (18) show that in the case of higher-order flexibility there is a multiple zero at $t_0 = 0$ for the determination of pairs of confocal second-order surfaces with the above mentioned property. Since this problem is algebraic of degree 8, a multiple zero of order > 8 at $t = 0$ implies already a one-parametric set of solutions, i.e., a continuously flexible octahedron.

4 Characterizing 2^{nd}- and 3^{rd}-order flexibility

The equations in Theorem 2 are necessary and sufficient for an octahedron $\mathbf{O}$ being infinitesmally flexible of order n. In order to obtain geometric characterizations for $n = 1, 2, \ldots$ we compare the coefficients of t^i, $i = 1, 2, \ldots$, in these equations. For this purpose we set up the Taylor expansions as

$$\begin{aligned} A(t) &= I_3 + A_1 t + A_2 t^2 + \ldots, & \mathbf{a}(t) &= \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \ldots, \\ \gamma(t) &= \gamma_1 t + \gamma_2 t^2 + \ldots, & \mathbf{b}(t) &= \mathbf{b}_1 t + \mathbf{b}_2 t^2 + \ldots. \end{aligned}$$

The inverse matrix $A^{-1}(t)$ can be expanded in the form

$$A^{-1}(t) = I_3 + B_1 t + B_2 t^2 + \ldots$$

with

$$\begin{aligned} B_1 &= -A_1 & B_3 &= -A_3 + A_2 A_1 + A_1 A_2 - A_1^3 \\ B_2 &= -A_2 + A_1^2 & B_k &= -A_k - A_{k-1} B_1 - \ \ldots \ - A_1 B_{k-1}. \end{aligned}$$

This implies for the flexion (11) according to (12) and (15)

$$\mathbf{p}_{i,r} = \mathbf{a}_i + A_i \mathbf{p}_i, \quad \mathbf{q}_{j,r} = \mathbf{b}_j + B_j^T \mathbf{q}_j \quad \text{for} \quad r = 1, \ldots, n. \tag{21}$$

The coefficients of t in (17), (19) and (18) yield

$$f_1(\mathbf{p}_i, \mathbf{p}_i) = f_1(\mathbf{p}_i, \mathbf{p}_{i+1}) = f_1(\mathbf{q}_j, \mathbf{q}_j) = 0 \quad \text{for} \quad i = 0, \ldots, 3 \quad \text{and} \quad j = 1, 2$$

with

$$f_1(\mathbf{x}, \mathbf{y}) := \mathbf{x}^T (A_1 + A_1^T) \mathbf{y} + (\mathbf{a}_1^T - \mathbf{b}_1^T)(\mathbf{x} + \mathbf{y}) - \gamma_1. \tag{22}$$

$\Phi : f_1(\mathbf{x}, \mathbf{x}) = 0$ is the second-order surface mentioned in statement (iv) of Theorem 1 (Fig. 1).

For the velocity vectors $\mathbf{p}_{i,1} = \mathbf{a}_1 + A_1\mathbf{p}_i$ and $\mathbf{q}_{j,1} = \mathbf{b}_1 - A_1^T\mathbf{q}_j$ according to (21) we can suppose a symmetric matrix A_1 and $\mathbf{b}_1 = -\mathbf{a}_1$. This is possible since otherwise we superimpose a motion which appoints to each point $\mathbf{x}$ the instantaneous velocity vector

$$\mathbf{v}(\mathbf{x}) := -\tfrac{1}{2}(\mathbf{a}_1 + \mathbf{b}_1) - \tfrac{1}{2}(A_1 - A_1^T)\mathbf{x}$$

with the skew symmetric matrix $(A_1 - A_1^T)$. Then we obtain the vectors

$$\text{(23)} \qquad \mathbf{p}_{i,1} = \tfrac{1}{2}(\mathbf{a}_1 - \mathbf{b}_1) + \tfrac{1}{2}(A_1 + A_1^T)\mathbf{p}_i\,, \quad \mathbf{q}_{j,1} = -\tfrac{1}{2}(\mathbf{a}_1 - \mathbf{b}_1) - \tfrac{1}{2}(A_1 + A_1^T)\mathbf{q}_j\,,$$

which still solve (5). These vectors are *orthogonal to* Φ (see Fig. 1) as according to (22) the equation of the polar plane of point $\mathbf{r}$ with respect to Φ reads

$$\text{(24)} \qquad f_1(\mathbf{x},\mathbf{r}) = \mathbf{x}^T\left[(A_1 + A_1^T)\mathbf{r} + (\mathbf{a}_1 - \mathbf{b}_1)\right] + (\mathbf{a}_1^T - \mathbf{b}_1^T)\mathbf{r} - \gamma_1 = 0\,.$$

The coefficients of t^2 in (17), (19) and (18) lead to the following equations: $f_2(\mathbf{p}_i,\mathbf{p}_i) = f_2(\mathbf{p}_i,\mathbf{p}_{i+1}) = 0$ for $i = 0,\dots,3$ with

$$\text{(25)} \qquad \begin{aligned} f_2(\mathbf{x},\mathbf{y}) :=\ & \mathbf{x}^T(A_2^T + A_1^TA_1 + A_2)\mathbf{y} + \\ & + \left[(\mathbf{a}_1^T - \mathbf{b}_1^T)A_1 + (\mathbf{a}_2^T - \mathbf{b}_2^T)\right](\mathbf{x}+\mathbf{y}) + (\mathbf{a}_1^T - \mathbf{b}_1^T)\mathbf{a}_1 - \gamma_2\,, \end{aligned}$$

and $g_2(\mathbf{q}_j,\mathbf{q}_j) = 0$, $j = 1,2$, with

$$\text{(26)} \qquad \begin{aligned} g_2(\mathbf{x},\mathbf{y}) :=\ & \mathbf{x}^T(A_2 - A_1A_1^T + A_2^T - A_1^2 - A_1^{T2})\mathbf{y} + \\ & + \left[-(\mathbf{a}_1^T - \mathbf{b}_1^T)A_1^T + (\mathbf{a}_2^T - \mathbf{b}_2^T)\right](\mathbf{x}+\mathbf{y}) + (\mathbf{a}_1^T - \mathbf{b}_1^T)\mathbf{b}_1 - \gamma_2\,. \end{aligned}$$

The difference

$$\text{(27)} \qquad \begin{aligned} h_2(\mathbf{x},\mathbf{y}) :=\ & f_2(\mathbf{x},\mathbf{y}) - g_2(\mathbf{x},\mathbf{y}) = \\ =\ & \mathbf{x}^T(A_1{+}A_1^T)^2\,\mathbf{y} + (\mathbf{a}_1^T{-}\mathbf{b}_1^T)(A_1{+}A_1^T)(\mathbf{x}{+}\mathbf{y}) + (\mathbf{a}_1^T{-}\mathbf{b}_1^T)(\mathbf{a}_1{-}\mathbf{b}_1) = \\ =\ & \left[\mathbf{x}^T(A_1{+}A_1^T) + (\mathbf{a}_1^T - \mathbf{b}_1^T)\right]\left[(A_1{+}A_1^T)\mathbf{y} + (\mathbf{a}_1 - \mathbf{b}_1)\right] \end{aligned}$$

depends only on the first derivatives $\mathbf{a}_1$, $\mathbf{b}_1$, A_1, but not γ_1.

$\Psi_p : f_2(\mathbf{x},\mathbf{x}) = 0$ and $\Psi_q : g_2(\mathbf{x},\mathbf{x}) = 0$ are two surfaces of second order, one passing through the sides of $\mathbf{p}_0 \dots \mathbf{p}_3$, the other through $\mathbf{q}_1$ and $\mathbf{q}_2$. The surface $\Psi : h_2(\mathbf{x},\mathbf{x}) = 0$ belongs to the pencil $[\Psi_p\Psi_q]$ spanned by Ψ_p and Ψ_q. According to the last line in (27) the equation $h_2(\mathbf{x},\mathbf{y}) = 0$ is equivalent to the statement that the points $\mathbf{x}$ and $\mathbf{y}$ have orthogonal polar planes with respect to Φ (note (24)). Hence $\Psi : h_2(\mathbf{x},\mathbf{x}) = 0$ is *polar to the absolute conic* with respect to Φ, provided Φ is regular.

In Table 1 all possible cases for Φ are listed. Here the equations (22) of Φ are given in standard form. The corresponding equations (27) of Ψ and of one particular surface Ψ' selected from the pencil $[\Phi\Psi]$ can be found in Table 2.

These specifications prove, that for a nontrivial flex the polynomial $h_2(\mathbf{x},\mathbf{x})$ is never proportional to $f_1(\mathbf{x},\mathbf{x})$, $f_2(\mathbf{x},\mathbf{x})$ or $g_2(\mathbf{x},\mathbf{x})$. Since for regular Φ the (imaginary) surface Ψ is polar to the absolute conic, the pencil $[\Phi\Psi]$ must be polar to the linear system of surfaces confocal to Φ. The cylinder Ψ' in the first two cases of Table 2 is polar to one focal conic of Φ with respect to Φ.

type of Φ	*reduced equ. of* Φ	$A_1 = A_1^T$	$\mathbf{a}_1 = -\mathbf{b}_1$	γ_1
one-sheet hyperb.	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	$\begin{pmatrix} 1/a^2 & 0 & 0 \\ 0 & 1/b^2 & 0 \\ 0 & 0 & -1/c^2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	2
hyp. paraboloid	$\frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0$	$\begin{pmatrix} 1/a^2 & 0 & 0 \\ 0 & -1/b^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}$	0
intersecting planes	$k^2x^2 - z^2 = 0$	$\begin{pmatrix} k^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	0

Table 1: Different cases of Φ

type of Φ	$h_2(\mathbf{x}, \mathbf{x})$	*surface* Ψ'
hyperboloid	$\frac{4x^2}{a^4} + \frac{4y^2}{b^4} + \frac{4z^2}{c^4}$	$\frac{a^2+c^2}{a^4}x^2 + \frac{b^2+c^2}{b^4}y^2 = 1$ ... ell. cylinder
hyp. paraboloid	$\frac{4x^2}{a^4} + \frac{4y^2}{b^4} + 1$	$\frac{a^2+b^2}{a^4}x^2 - z + \frac{b^2}{4} = 0$ ... parab. cylinder
inters. planes	$4k^4x^2 + 4z^2$	$z^2 = 0$ ... two-fold plane of symmetry

Table 2: Different cases of Ψ and Ψ'

For a 2[nd]-order flex of $\mathbf{O}$ obeying (22) it is necessary that there is a surface Ψ_p passing through the sides of the skew quadrangle $\mathbf{p}_0 \dots \mathbf{p}_3$ and a surface Ψ_q through $\mathbf{q}_1$ and $\mathbf{q}_2$ such that the pencil $[\Psi_p\Psi_q]$ spanned by Ψ_p and Ψ_q contains Ψ. With Φ and Ψ_p each surface Ψ'_p in the pencil $[\Phi\Psi_p]$ passes through the equator $\mathbf{p}_0 \dots \mathbf{p}_3$. In the same way each $\Psi'_q \in [\Phi\Psi_q]$ contains $\mathbf{q}_1$ and $\mathbf{q}_2$. And any $\Psi'_p \in [\Phi\Psi_p] \setminus \{\Phi\}$ and $\Psi'_q \in [\Phi\Psi_q] \setminus \{\Phi\}$ span a pencil which contains any $\Psi' \in [\Phi\Psi]$ since all these surfaces are contained in the two-dimensional linear system spanned by Φ, Ψ and Ψ_p (see Fig. 4 where the linear system is represented as a projective plane).

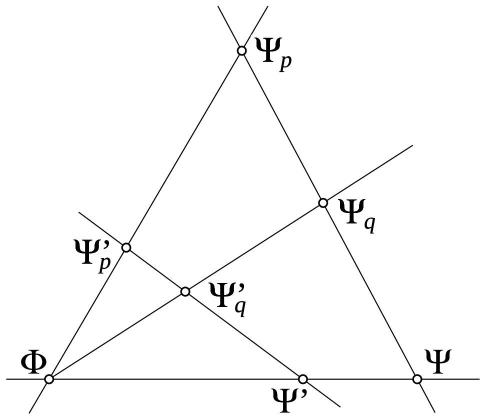

Figure 2: The linear system of second-order surfaces spanned by Φ, Ψ and Ψ_p

Conversely, if a pair (Ψ'_p, Ψ'_q) of second-order surfaces with Ψ'_p passing through the equator and $\mathbf{q}_1, \mathbf{q}_2 \in \Psi'_q$ is given such that the pencil $[\Psi'_p\Psi'_q]$ contains any $\Psi' \in [\Phi, \Psi] \setminus \{\Phi\}$, then the octahedron is infinitesimally flexible of order 2. This results from the following arguments: We can determine $\Psi_p \in [\Phi\Psi'_p]$ and $\Psi_q \in [\Phi\Psi'_q]$ such that the difference of their equations gives $h_2(\mathbf{x}, \mathbf{x}) = 0$ in (27). Then from the equations of Φ, Ψ_p and Ψ_q one gets $A_2 + A_2^T$ and

$\mathbf{a}_2 - \mathbf{b}_2$.

Also for the acceleration vectors $2\mathbf{p}_{i,2}$ and $2\mathbf{q}_{j,2}$ of the vertices we can assume a symmetric matrix A_2 and $\mathbf{a}_2 = -\mathbf{b}_2$ in (21)

$$\mathbf{p}_{i,2} = \mathbf{a}_2 + A_2\,\mathbf{p}_i \quad \text{and} \quad \mathbf{q}_{j,2} = \mathbf{b}_2 + B_2^T\mathbf{q}_j = \mathbf{b}_2 - (A_2^T - A_1^{T2})\mathbf{q}_j\,,$$

because otherwise according to (9) we could add at each vertex $\mathbf{x}$ the vector

$$\mathbf{a}(\mathbf{x}) := -\tfrac{1}{2}(\mathbf{a}_2 + \mathbf{b}_2) - \tfrac{1}{2}(A_2 - A_2^T)\mathbf{x}$$

and obtain the new solution of the system (6)

$$(28) \quad \mathbf{p}_{i,2} = \tfrac{1}{2}(\mathbf{a}_2 - \mathbf{b}_2) + \tfrac{1}{2}(A_2 + A_2^T)\mathbf{p}_i\,, \quad \mathbf{q}_{j,2} = -\tfrac{1}{2}(\mathbf{a}_2 - \mathbf{b}_2) - \tfrac{1}{2}(A_2 + A_2^T - 2A_1^{T2})\mathbf{q}_j\,.$$

We demonstrate this method at the following

Example: Let Φ be a one-sheet hyperboloid according to Table 1, i.e.,

$$\Phi\colon\ f_1(\mathbf{x},\mathbf{x}) = x^2 + \frac{y^2}{2} - z^2 - 1 = 0 \quad \Longrightarrow \quad \Psi\colon\ h_2(\mathbf{x},\mathbf{x}) = x^2 + \frac{y^2}{4} + z^2 = 0\,.$$

We specify Ψ_p' as a pair of tangent planes of Φ and Ψ_q' as a hyperboloid:

$$\Psi_p'\colon\ y^2 - (z+1)^2 = 0 \quad \text{and} \quad \Psi_q'\colon\ 2x^2 + \frac{27y^2}{4} - 6(z+1)^2 - 1 = 0\,.$$

Then the pencil $[\Psi_p'\Psi_q']$ contains the associated elliptic cylinder $\Psi'\colon 2x^2 + \frac{3}{4}y^2 - 1 = 0$ from Table 2. Now we replace Ψ_p' by $\Psi_p \in [\Phi\Psi_p']$ and set $\Psi_q = \Psi_q'$ such that the difference of the equations $f_2(\mathbf{x},\mathbf{x}) = 0$ of Ψ_p and $g_2(\mathbf{x},\mathbf{x}) = 0$ of Ψ_q gives exactly the equation $h_2(\mathbf{x},\mathbf{x}) = 0$ of Ψ according to (27). This yields

$$f_2(\mathbf{x},\mathbf{x}) = -4x^2 - 26y^2 + 28z^2 + 48z + 28 \quad \text{and} \quad g_2(\mathbf{x},\mathbf{x}) = -8x^2 - 27y^2 + 24z^2 + 48z + 28\,.$$

The comparison with (22), (25) and (26) gives

$$A_1 = A_1^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_2 + A_2^T = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -\frac{105}{4} & 0 \\ 0 & 0 & 27 \end{pmatrix},$$

$$\mathbf{a}_1^T = -\mathbf{b}_1^T = (0,0,0), \quad \mathbf{a}_2^T - \mathbf{b}_2^T = (0,0,24), \quad \gamma_1 = 2\,, \quad \gamma_2 = -28\,.$$

The coordinates of the four vertices $\mathbf{p}_0, \dots, \mathbf{p}_3$ defining the quadrangle $\Phi \cap \Psi_p' = \Phi \cap \Psi_p$ and those of the poles $\mathbf{q}_1, \mathbf{q}_2$ selected from $\Phi \cap \Psi_q' = \Phi \cap \Psi_q$ are listed in Table 3 as well as the corresponding velocity vectors (23) and acceleration vectors (28) obeying the linear systems (5) and (6).

We summarize:

vertex	*velocity vector*	*half acceleration vector*
$\mathbf{p}_0^T = (0,2,1)$	$\mathbf{p}_{0,1}^T = (0,1,-1)$	$\mathbf{p}_{0,2}^T = (0,-\frac{105}{4},\frac{51}{2})$
$\mathbf{p}_1^T = (\sqrt{2},0,-1)$	$\mathbf{p}_{1,1}^T = (\sqrt{2},0,1)$	$\mathbf{p}_{1,2}^T = (-\frac{5}{2}\sqrt{2},0,-\frac{3}{2})$
$\mathbf{p}_2^T = (0,-2,1)$	$\mathbf{p}_{2,1}^T = (0,-1,-1)$	$\mathbf{p}_{2,2}^T = (0,\frac{105}{4},\frac{51}{2})$
$\mathbf{p}_3^T = (-\sqrt{2},0,-1)$	$\mathbf{p}_{3,1}^T = (-\sqrt{2},0,1)$	$\mathbf{p}_{3,2}^T = (\frac{5}{2}\sqrt{2},0,-\frac{3}{2})$
$\mathbf{q}_1^T = (\frac{1}{2}\sqrt{5},0,-\frac{1}{2})$	$\mathbf{q}_{1,1}^T = (-\frac{1}{2}\sqrt{5},0,-\frac{1}{2})$	$\mathbf{q}_{1,2}^T = (\frac{7}{4}\sqrt{5},0,-\frac{23}{4})$
$\mathbf{q}_2^T = (c,\sqrt{2},-c)$	$\mathbf{q}_{2,1}^T = (-c,-\frac{1}{2}\sqrt{2},-c)$	$\mathbf{q}_{2,2}^T = (\frac{7}{2}c,\frac{107}{8}\sqrt{2},-12+\frac{25}{2}c)$

$$c := 3 + \tfrac{1}{2}\sqrt{62} \quad \text{obeying} \quad 8c^2 - 24c - 13 = 0\,.$$

Table 3: Example of a 2^{nd}-order flexible octahedron

Theorem 3: *A first-order infinitesimally flexible octahedron* $\mathbf{O}$ *with vertices* $\mathbf{p}_0,\dots,\mathbf{q}_2 \in \Phi$ *and a non-coplanar equator* $\mathbf{p}_0\dots\mathbf{p}_3$ *is infinitesimally flexible of order two if and only if there are second-order surfaces* $\Psi'_p \neq \Phi$ *through the sides of the equator and* $\Psi'_q \neq \Phi$ *through the poles* $\mathbf{q}_1,\mathbf{q}_2$ *such that the pencil spanned by* Ψ'_p *and* Ψ'_q *shares a surface* Ψ' *with the pencil spanned by* Φ *and the associated* $\Psi : h_2(\mathbf{x},\mathbf{x}) = 0$ *as listed in Table 2.*

Corollary 1: *A second-order flexible octahedron* $\mathbf{O}$ *can be built from five arbitrary vertices* $\mathbf{p}_0,\dots,\mathbf{q}_1$*, provided* $\{\mathbf{p}_0,\dots,\mathbf{p}_3\}$ *are not coplanar. There is a free choice for the last vertex* $\mathbf{q}_2$ *on a space-curve of order four passing through* $\mathbf{q}_1$*.*

Proof: The equator $\mathbf{p}_0\dots\mathbf{p}_3$ and the pole $\mathbf{q}_1$ define the surface Φ uniquely. Then we specify any other second-order surface Ψ'_p through the equator. There is one surface $\Psi'_q \in [\Psi'_p\Psi]$ passing through $\mathbf{q}_1$. In order to meet the conditions of Theorem 3, it is sufficient to specify pole $\mathbf{q}_2$ on the curve of intersection between Φ and Ψ'_q. $\square$

Remark: Second-order flexibility of any framework means that to each vertex we can assign a curvature center in a way which is compatible with the given edges (see [13]). In this sense there is also a more kinematic characterization for 2^{nd}-order flexibility of octahedra based on the relative motion between opposite faces[4] as displayed in Fig. 4: An octahedron $\mathbf{O}$ is flexible of order two if and only if there is a spatial motion such that the coplanar lines $\mathbf{q}_1\mathbf{r}_1, \mathbf{r}_1\mathbf{p}_1, \mathbf{p}_1\mathbf{q}_1$ can serve as curvature axes for the trajectories of the moving points $\mathbf{p}_2$, $\mathbf{q}_2$, $\mathbf{r}_2$, respectively. The relation between movings points and their instantaneous curvature axes under a spatial motion is e.g. treated in [3], p. 169 ff.

The coefficients of t^3 in (17) and (18) give rise to the equations $f_3(\mathbf{p}_i,\mathbf{p}_i) = 0\,,\ i = 0,\dots,3\,,$

[4]These motions play also a role in robotics: They are unexpected infinitesimal or even finite self-motions at "singular postures" of particular parallel manipulators (see [7] or [8]).

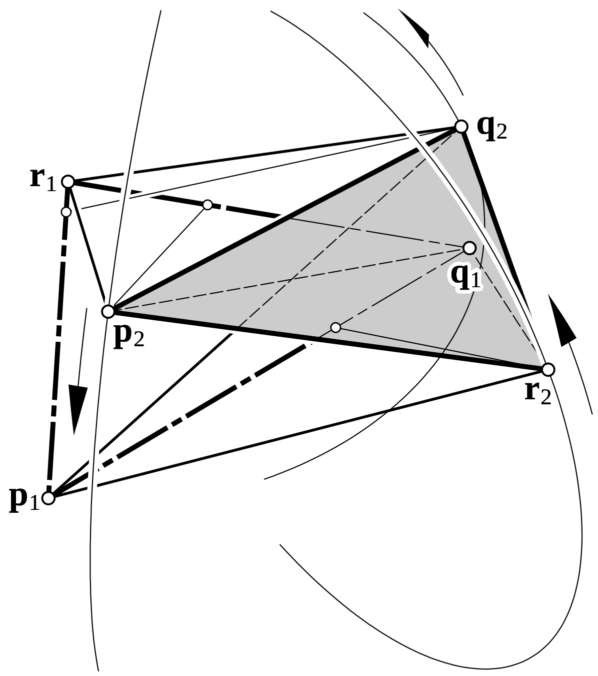

Figure 4: The relative motion between opposite faces $\mathbf{p}_1\mathbf{q}_1\mathbf{r}_1$ and $\mathbf{p}_2\mathbf{q}_2\mathbf{r}_2$ of a 2nd-order flexible octahedron $\mathbf{O}$: The curvature circles of the moving vertices $\mathbf{p}_2, \mathbf{q}_2, \mathbf{r}_2$ have coplanar axes $\mathbf{q}_1\mathbf{r}_1$, $\mathbf{r}_1\mathbf{p}_1$, $\mathbf{p}_1\mathbf{q}_1$, respectively.

and $g_3(\mathbf{q}_j, \mathbf{q}_j) = 0$, $j = 1, 2$, with bilinear functions

$$\begin{aligned} f_3(\mathbf{x},\mathbf{y}) := \; & \mathbf{x}^T(A_3^T + A_2^T A_1 + A_1^T A_2 + A_3)\mathbf{y} + \Big[(\mathbf{a}_1^T - \mathbf{b}_1^T)A_2 + (\mathbf{a}_2^T - \mathbf{b}_2^T)A_1 + \\ & +(\mathbf{a}_3^T - \mathbf{b}_3^T)\Big](\mathbf{x}+\mathbf{y}) + \Big[(\mathbf{a}_1^T - \mathbf{b}_1^T)\mathbf{a}_2 + (\mathbf{a}_2^T - \mathbf{b}_2^T)\mathbf{a}_1\Big] - \gamma_3 \end{aligned} \tag{29}$$

$$\begin{aligned} g_3(\mathbf{x},\mathbf{y}) := \; & \mathbf{x}^T(A_3 - A_2A_1 - A_1A_2 + A_1^3 - A_2A_1^T + A_1^2A_1^T - A_1A_2^T + \\ & +A_1A_1^{T2} + A_3^T - A_1^TA_2^T - A_2^TA_1^T + A_1^{T3})\mathbf{y} + \Big[(\mathbf{a}_1^T - \mathbf{b}_1^T)(-A_2^T + A_1^{T2}) - \\ & -(\mathbf{a}_2^T - \mathbf{b}_2^T)A_1^T + (\mathbf{a}_3^T - \mathbf{b}_3^T)\Big](\mathbf{x}+\mathbf{y}) + \Big[(\mathbf{a}_1^T - \mathbf{b}_1^T)\mathbf{b}_2 + (\mathbf{a}_2^T - \mathbf{b}_2^T)\mathbf{b}_1\Big] - \gamma_3 \end{aligned} \tag{30}$$

Again, these define surfaces of second order, one passing through the sides of $\mathbf{p}_0 \ldots \mathbf{p}_3$, the other through the poles $\mathbf{q}_1, \mathbf{q}_2$. The difference of these functions

$$\begin{aligned} h_3(\mathbf{x},\mathbf{y}) := \; & \mathbf{x}^T\Big[(A_2 - A_1^2 + A_2^T)(A_1 + A_1^T) + (A_1 + A_1^T)(A_2 - A_1^{T2} + A_2^T)\Big]\mathbf{y} + \\ & + \Big[(\mathbf{a}_1^T - \mathbf{b}_1^T)(A_2 - A_1^{T2} + A_2^T) + (\mathbf{a}_2^T - \mathbf{b}_2^T)(A_1 + A_1^T)\Big](\mathbf{x}+\mathbf{y}) + \\ & +2(\mathbf{a}_1^T - \mathbf{b}_1^T)(\mathbf{a}_2 - \mathbf{b}_2) = 0 \end{aligned} \tag{31}$$

depends only on the first and second derivatives of $\mathbf{a}(t)$, $\mathbf{b}(t)$ and $A(t)$ at $t = 0$, hence on the bilinear functions f_1, f_2 and g_2 according to (22), (25) and (26).

A geometric interpretation for these conditions in analogy to Theorem 3 is based on the surfaces Ω: $h_3(\mathbf{x},\mathbf{x}) = 0$, Ω_p: $f_3(\mathbf{x},\mathbf{x}) = 0$ and Ω_q: $g_3(\mathbf{x},\mathbf{x}) = 0$. This yields

Theorem 4: *A second-order flexible octahedron* $\mathbf{O}$ *with non-coplanar equator* $\mathbf{p}_0 \ldots \mathbf{p}_3$ *and poles* $\mathbf{q}_1, \mathbf{q}_2$ *according to Theorem 3 is infinitesimally flexible of order three if and only if* *there are surfaces $\Omega'_p \neq \Phi$ through the sides of $\mathbf{p}_0 \dots \mathbf{p}_3$ and $\Omega'_q \neq \Phi$ through $\mathbf{q}_1, \mathbf{q}_2$ such that the pencil spanned by Ω'_p and Ω'_q shares a surface Ω' with the pencil spanned by Φ and $\Omega\colon h_3(\mathbf{x},\mathbf{x}) = 0$, associated to Φ, Ψ_p and Ψ_q.*

The quadric Ω is defined by Φ, Ψ_p and Ψ_q. But the geometric meaning of this dependence has not been figured out yet.

Corollary 2: *A third-order flexible octahedron $\mathbf{O}$ can be built from five arbitrary vertices $\mathbf{p}_0, \dots, \mathbf{q}_1$, provided $\{\mathbf{p}_0, \dots, \mathbf{p}_3\}$ are not coplanar. The last vertex $\mathbf{q}_2$ is a point of intersection between the three quadrics Φ, Ψ'_q, Ω'_q passing through $\mathbf{q}_1$.*

Proof: In addition to the choice used in the proof of Corollary 1 we set $\Omega'_p = \Psi'_p$ and specify $\Omega'_q \in [\Omega'_p \Omega]$ as the surface passing through $\mathbf{q}_1$. $\square$

Suppose $\mathbf{O}$ is of BRICARD's type 1 or 2. Then the quadrangle $\mathbf{p}_0, \dots, \mathbf{p}_3$ must be symmetric with respect to a plane or line. For type 3 the quadric Φ must be a hyperboloid of revolution (see e.g. [12]). According to Corollary 2 there are third-order flexible octahedra which do not obey any of these necessary conditions. This reveals that for octahedra infinitesimal flexibility of order 3 does not imply continous flexibility.

Example: For the data given above in Table 3 we get

$$\Omega\colon\ h_3(\mathbf{x},\mathbf{x}) = -24x^2 - 53y^2 - 104z^2 - 96z = 0\,, \quad \Omega'_q\colon\ 24x^2 + 85y^2 + 72z^2 + 32z - 32 = 0\,.$$

The only real solutions for $\mathbf{q}_2$ are $\left(\pm\frac{1}{2}\sqrt{5}, 0, -\frac{1}{2}\right)$. The solution $\mathbf{q}_2 \neq \mathbf{q}_1$ gives a BRICARD octahedron of types 1 and 2, simultaneously. The example presented in Table 3 with a different choice of $\mathbf{q}_2$ is exactly of 2^{nd}-order flexibility.

There are analogous conditions for infinitesimal flexibility of order 4 and higher. In the sense of Fig. 4 these conditions express projective dependencies of particular quadrics in the projective 9-space of quadrics in $\mathbb{E}^3$.

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