Rigidity of Certain Polyhedra in $\mathbf{R}^3$

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The rigidity theorem of Cauchy states that a convex polyhedron in $\mathbf{R}^3$ is rigid among convex polyhedra. The example of a house with its roof caved in shows that there is a polyhedron isomorphic to a convex polyhedron but not congruent to it in $\mathbf{R}^3$ (figure 1). We will see that the example in figure 2 is locally rigid and globally rigid among polyhedra with the same generalized Gauss map.

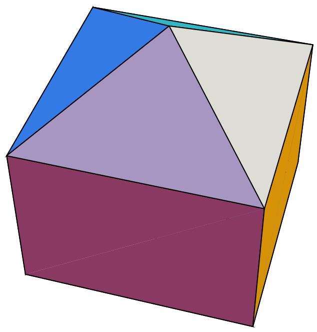

Figure 1. Convex roof.

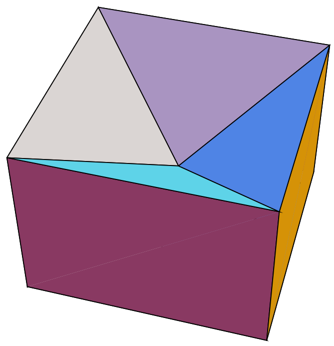

Figure 2. Sunken roof.

J.J. Stoker and I.Kh. Sabitov extended Cauchy's rigidity theorem to some non-convex spherical polyhedra, and H. Gluck proved that almost every simply connected polyhedron is locally rigid [**9**], [**5**], [**4**]. We will obtain rigidity of certain polyhedra parametrized by the sphere and the projective plane; e.g. the heptahedron is rigid [**6**]. We know no previous example of a rigid projective plane.

The first example of a flexible spherical polyhedron appears to have been discovered in 1897 by Raoul Bricard (we define this in the next section and describe the example in section 7). It is modeled on an octahedron. In 1977, Robert Connelly created an embedded flexible spherical polyhedron [**2**]. We refer the reader to Connelly's excellent survey paper [**3**].

Cauchy proved his theorem by considering the geodesic polygon associated to a vertex v of M obtained by intersecting M with a small sphere centered at v. When M is convex, this gives a convex geodesic polygon (the *link* of v) and Cauchy used these polygons in a brilliant manner to prove rigidity of convex polyhedra. We describe Cauchy's proof in section 3.

When M is not convex at v, the link of v is not convex on the sphere. However the "Gaussian image" of v may very well be an embedded convex geodesic polygon on the sphere (we make this precise in the sequel). For example, a saddle point as in figure 4, or a roof vertex as in figure 5. To obtain our rigidity theorem, we will use these convex geodesic polygons in the same way Cauchy used the links of convex vertices. This idea already appears in [**9**] for simple saddle vertices. The novelty of our approach is that the Gauss maps we use come from orientations of the faces of M which need not define an orientation (even local) of M. This is natural from the herisson point of view (cf. Remark 4). Since our approach uses only face orientations, we are able to obtain rigidity results for the projective plane.

For example, a polyhedral sphere or projective plane whose vertices are convex, simple saddle and cones on a figure eight is ε-rigid (this is an application of theorem 2). In theorem 6, we prove a global rigidity theorem.

1. Preliminaries.

Let Σ denote a simplicial complex homeomorphic to the sphere or projective plane. We say M is a *polyhedron* in $\mathbf{R}^3$, modeled on Σ, if M is the image of Σ by a continuous map f, isometric on each face of Σ, and the image of each face of Σ is a convex polygon in $\mathbf{R}^3$. We assume that f is injective on the star of each vertex v or that f has a stable singularity at v, that is, M is combinatorially a cone on a figure eight at v; cf. figure 6. The figure eight is assumed to have four edges. We will call a vertex that is a cone on a figure eight, a *figure eight vertex*.

We say a polyhedron N, modeled on Σ, is *ε-close* to M, if N is the image of Σ by a map $\tilde{f}$ and for each face F of Σ, $f(F)$ is ε-close to $\tilde{f}(F)$. Since f and $\tilde{f}$ are isometric on each face, N is ε-close to M if $f(v)$ and $\tilde{f}(v)$ are ε-close in $\mathbf{R}^3$, for each vertex v of Σ. When $f(F)$ and $\tilde{f}(F)$ are congruent in $\mathbf{R}^3$ for each face F of Σ, we say M is *isomorphic* to N.

When there is no danger of confusion, we will identify a face F of Σ with its image $f(F)$ in M, so this face of Σ will have sides of a fixed length and interior angles between the sides of a fixed size.

2. A Gauss Map.

Now let M be a polyhedron in $\mathbf{R}^3$ parametrized by the sphere or projective plane. For each face F of M, we choose a unit vector $z(F) \in S$ (the unit 2-sphere), orthogonal to F, called an orientation of F. Even if M is orientable, we do not assume the $z(F)$ orient M. When F_1 and F_2 have an edge ℓ in common, we define the *dihedral angle* between the faces at ℓ to be the length of the shortest geodesic on S joining $z(F_1)$ to $z(F_2)$; we always assume the dihedral angles are strictly between 0 and π.

At a vertex v of M, let $F_1, \ldots, F_k$ be the faces in the star of v (thought of in the domain Σ), labeled so that we turn once around v on M (which way is not important). Define $g(v)$ to be the geodesic polygon on S, consisting of the shortest geodesic arcs joining $z(F_1)$ to $z(F_2)$, $z(F_2)$ to $z(F_3), \ldots, z(F_k)$ to $z(F_1)$. We call $g(v)$ a *Gaussian image of v*.

We define the vertex v to be *simple* if $g(v)$ is an embedded convex geodesic polygon on S.

We will see later that if it is possible to orient the faces of M so that each vertex is simple, then M is *ε-rigid*; i.e., if N is ε-close to M and isomorphic to M then M is congruent to N.

Before proving this, we discuss some examples of simple vertices. Each vertex v of a strictly convex polyhedron is simple for an orientation of each face coming from the outward orientation of M. In this case, there is a nice relation between the link of v and the Gaussian image $g(v)$: they are dual on S. The *link of* v is the intersection of M with a small round sphere centered at v, rescaled to obtain a geodesic polygon $\ell k(v)$, on S.

The length of a side of $\ell k(v)$, coming from the intersection of a face F at v is the angle of F at v. The angle formed by two sides of $\ell k(v)$ is the dihedral angle between the two faces, cf. Figure 3.

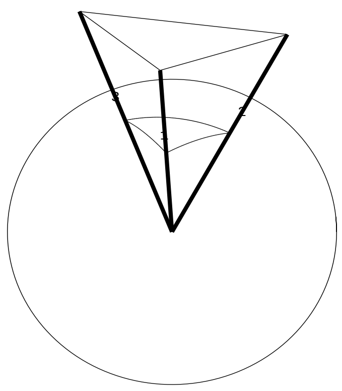

Figure 3. Link of a convex vertex

The *dual* of a convex polygon Γ on S is the convex polygon C on S obtained from Γ by going a distance $\pi/2$ along the geodesic normal to Γ and exiting from the convex domain of S bounded by Γ. Each side of length ℓ of Γ goes to a vertex of C of angle $\pi - \ell$, and each vertex of Γ of angle θ goes to a side of C of length $\pi - \theta$.

The reader can now verify that $g(v)$ is the dual of $\ell k(v)$, at a strictly convex vertex v. In general however, $g(v)$ may be embedded and convex and not $\ell k(v)$.

Our next example of a simple vertex is a symmetric saddle type vertex as in figure 4. Now $\ell k(v)$ is not convex, but $g(v)$ is convex. Here we choose an orientation of each face coming from a global orientation of the star of v (the neighborhood of v which is the union of the four faces at v). The length of a side of $g(v)$ is the dihedral angle between the faces determining the side. The angle between two sides of $g(v)$ at $g(F) = z(F)$ (F a face at v) is the angle of F at v.

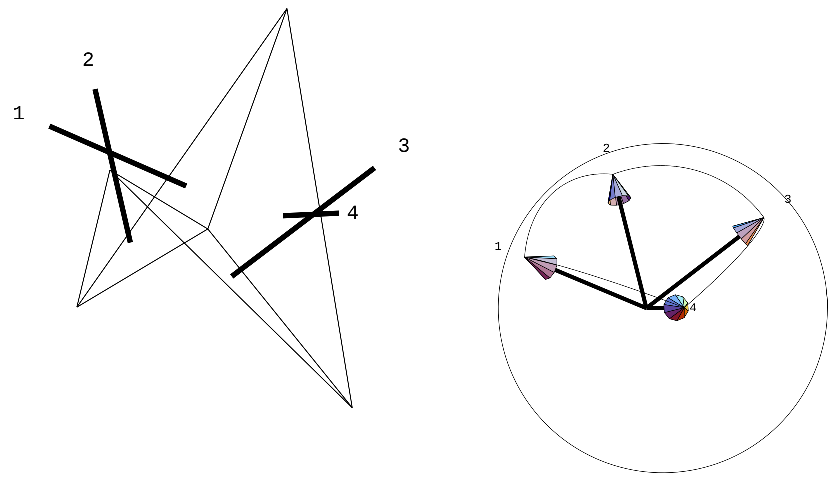

Figure 4. A simple saddle and its Gauss image

The third example of a simple vertex is the "fold" vertex arising from the caved in roof of figure 2. There are four faces at v. The two "wall" faces F_1, F_2 are oriented with the normal directions pointing out of the house, and the other two faces of the sunken roof, F_3, F_4, are oppositely oriented, i.e., their normals orient $F_3 \cup F_4$ but not the star of v. The lengths of the sides of $g(v)$ are the dihedral angles (of this orientation) and the angles between the sides are α_1, α_2, $\pi - \alpha_3$, and $\pi - \alpha_4$, where α_i is the angle of F_i at v.

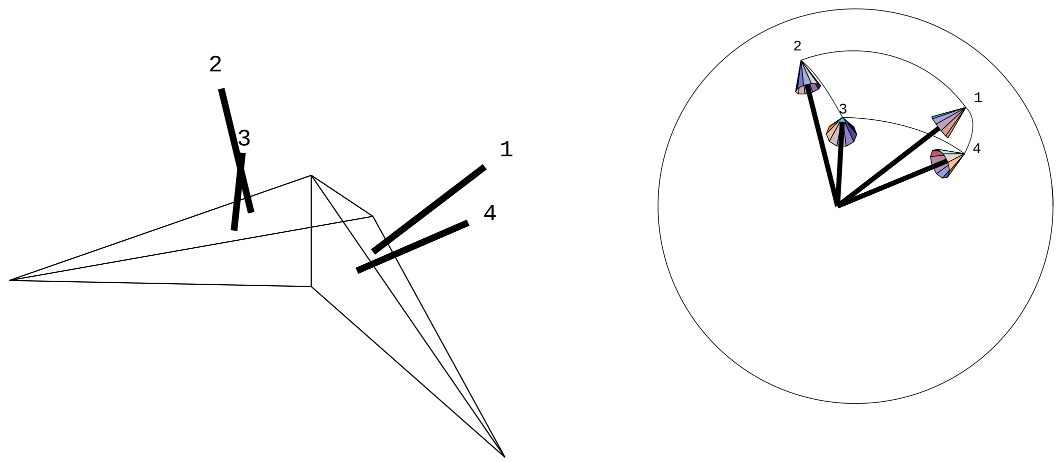

Figure 5. A fold and its Gauss image

A last example here of a simple vertex is a cone on a figure eight. In figure 6 we illustrate a face orientation of a cone on a figure 8 and its gauss image.

3. Local Rigidity.

We now state one of our local results, for a polyhedron M modeled on Σ (the sphere or projective plane).

THEOREM 1. *Assume each face of M can be oriented so that each vertex of M is a simple vertex, for the Gauss map arising from the face orientations. Then M is ε-rigid, i.e., there is a $\varepsilon > 0$ such that if N is a polyhedron ε-close to M, and if N*

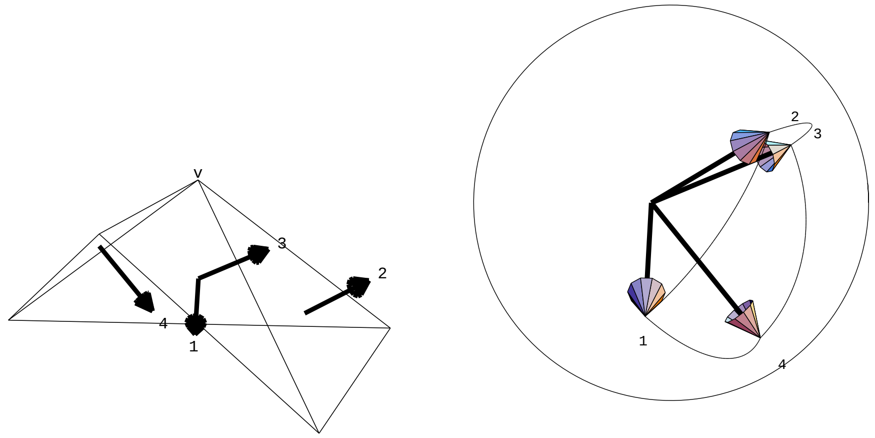

Figure 6. A figure 8 and its Gauss image

is isomorphic to M, then N is congruent to M.

The proof of this theorem is similar to Cauchy's proof of the rigidity of convex polyhedra. To each edge of M we will associate a $+$ or $-$ (or nothing) depending on whether the dihedral angle of the corresponding edge of N is larger, or smaller (or equal) to that of M. Then we prove that for each vertex v of M, the total number of sign changes of the edges at v, from $+$ to $-$ or $-$ to $+$, is zero or at least 4, when one goes around the vertex v of M exactly once (this is called Cauchy-Steinitz lemma, or Cauchy's geometric lemma below). A combinatorial argument then proves that this number must always be zero for each vertex. Thus no edge is marked, all the dihedral angles are equal and M is congruent to N.

Cauchy's Geometric lemma, also called the Cauchy-Steinitz lemma. *Let Γ_1 and Γ_2 be embedded convex spherical or planar polygons. Assume their sides are in bijective correspondence when they are traversed in the same sense and the lengths of corresponding sides are equal. The corresponding interior angles are compared in magnitude, and on Γ_1, each interior angle is marked with a plus or minus (or nothing) depending on whether it is larger or smaller (or equal) to the* *corresponding interior angle of* Γ_2.

Define the index of Γ_1, *Ind*(Γ_1), *to be the number of sign changes at the interior angles of* Γ_1 *(plus to minus or minus to plus) as one turns around* Γ_1 *once.*

Then Ind$(\Gamma_1) = 0$, *or Ind*$(\Gamma_1) \geq 4$.

The reader can refer to [**9**] for a discussion and proof of Cauchy's geometric lemma.

The dual of an embedded spherical convex polygon is also an embedded convex polygon. Sides go to vertices (with the length going to the interior angle which is the complement of the length) and vertices go to sides (with the interior angle becoming a length which is the complement of the angle). Thus one has a dual Cauchy lemma with the role of the side lengths and interior angles interchanged.

Cauchy's Dual Geometric lemma. *Let* Γ_1 *and* Γ_2 *be convex spherical polygons as above where now it is assumed that the corresponding interior angles are equal. Each edge of* Γ_1 *is assigned a plus or minus (or nothing) depending on how its length compares to the corresponding edge of* Γ_2. *An index, Ind*(Γ_1) *is defined, which counts the number of sign changes of the edges as one traverses* Γ_1 *once. Then Ind*(Γ_1) *is zero or at least four.*

Clearly, in both of these lemmas, if the index is zero, then the polygons are congruent.

We will use a Cauchy type lemma for spherical figure eights.

The figure eight lemma. *Let* Γ_1 *and* Γ_2 *be non-embedded polygonal 4-gons on the sphere or plane (i.e., they are figure eights). Assume that the lengths of corresponding sides of* Γ_1 *and* Γ_2 *are equal when one traverses* Γ_1 *and* Γ_2 *in the same sense, and that they are less than* π *if the polygons are spherical. The interior angles are compared in magnitude and each interior angle of* Γ_1 *is given a plus or minus (or nothing), and an index is defined for* Γ_1 *as expected. Then* Γ_1 *and* Γ_2 *are congruent or Ind*$(\Gamma_1) = 4$.

PROOF: For spherical figure eights, this is a corollary of Cauchy's geometric lemma. To see this let a_i , b_i , c_i , d_i be the vertices of Γ_i , $i = 1, 2$. Let α_i , β_i , γ_i , δ_i be the interior angles at these vertices. Suppose $[a_i, b_i]$ and $[c_i, d_i]$ are the edges of Γ_i that cross each other. Then $[a_i, -b_i, -c_i, d_i]$ is a convex quadrilateral Γ'_i whose angles are $\pi - \alpha_i$, $\pi - \beta_i$, $\pi - \gamma_i$, and $\pi - \delta_i$. Then Cauchy's geometric lemma applies to Γ'_1 and Γ'_2 to give the result desired for Γ_1 and Γ_2 .

For planar figure eights we know of no such easy argument. In fact we will not use the planar figure eight lemma in this paper, however we include a proof of the lemma. The spherical figure eight lemma is used in remarks 3 and 5. Let Γ be a planar figure eight. We fix one segment of Γ to be the segment $I = [0, 1]$, and we assume I intersects the interior of one of the other segments of Γ, so that among the four vertices p, q, 0 and 1 of Γ, we can assume the segment $[p, q]$ of Γ, meets I, and q is in the upper half plane $\{y > 0\}$, and p in the lower half-plane; cf. figure 7.

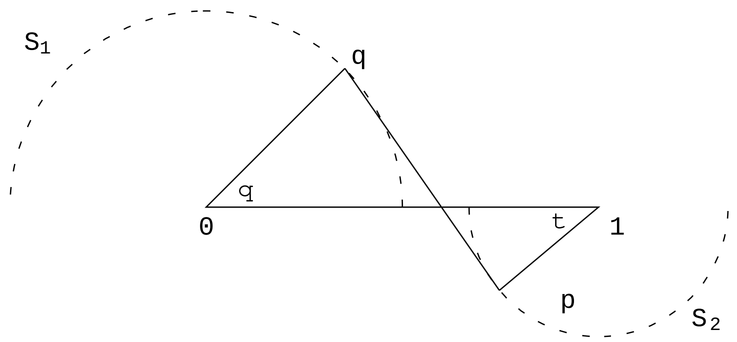

Figure 7.

Let θ and τ denote the inner angles at 0 and 1 respectively. The segment I will always remain fixed during the variations of Γ that follow.

First we will show that if Γ_1 is another figure eight like Γ, with corresponding sides of equal lengths, then one can not have $\theta_1 \geq \theta$, $\tau_1 \geq \tau$, with one of these a strict inequality (“like Γ” means, I is a segment of Γ_1 and the segment $[p_1, q_1]$ of Γ_1, meets I).

Let S_1 denote the semicircle centered at 0, of radius $||q||$, and in the upper half plane. Similarly, let S_2 be the semicircle in the lower half-plane, centered at 1 and of radius $||p-1||$. Then if $\theta_1 \geq \theta$, and $\tau_1 \geq \tau$, the point q_1 is on S_1 and to the left of q, and the point p_1 is on S_2, to the right of p.

Let $\tilde{q}$ be the point of S_1 on the line joining 0 to p, and let $q(t)$ be a parametrization of the arc on S_1 joining q to $\tilde{q}$, so that $q(0) = q$, $q(1) = \tilde{q}$. Notice that for each t, $0 \leq t < 1$, the points $q(s)$, $t < s \leq 1$, are outside the circle centered at p, of radius $||p - q(t)||$; cf. figure 8. In particular, $||q(s) - p|| > ||q(t) - p||$. Here we are using the following fact. Let S be a circle with center a and let p be a point different from a. The function distance from p to S has exactly two critical points, which are the two points of intersection of the line joining a to p with S.

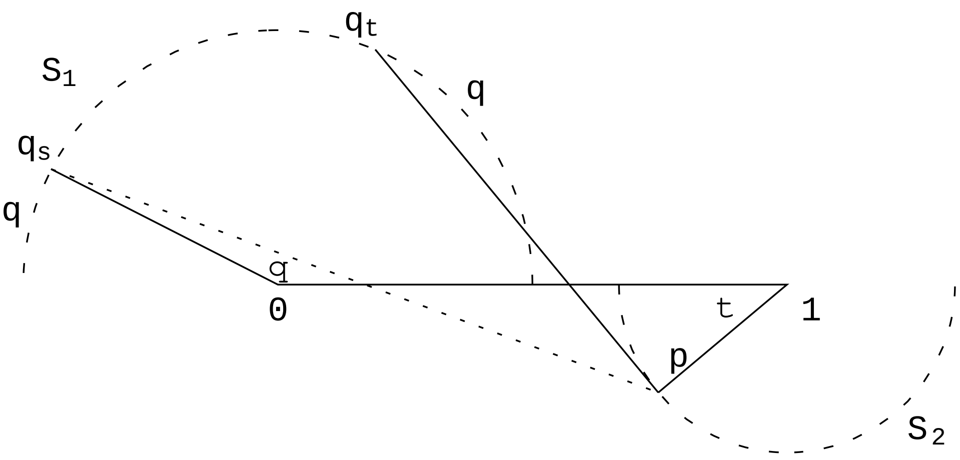

Figure 8.

Now consider the two figure eights Γ and Γ_1 with vertices $\{0, 1, p, q\}$ and $\{0, 1, p_1, q_1\}$ respectively. We want to show that if $\theta_1 \geq \theta$, $\tau_1 \geq \tau$, with one a strict inequality, then $||p_1 - q_1|| > ||p - q||$.

First move q to the left on S_1 until reaching either q_1 or $\tilde{q}$. We know this increases the length of the segment $[p, q]$. If one reaches $\tilde{q}$ before reaching q_1, then fixing $\tilde{q}$, begin moving p along S_2, to the right until either reaching p_1 or until reaching $\tilde{p}$, where $\tilde{p}$ is the point on S_2 where 1 is on the segment $[\tilde{q}, \tilde{p}]$. Again the length of the segment will be increasing. Note that if the inner angle at p is more then $\pi/2$, then the quadrilateral will cease to be a figure eight when one moves to the right (and becomes a figure eight at a later point), but this does not matter for us. This happens as p moves to the right from the position in 8 to the one in figure 9.

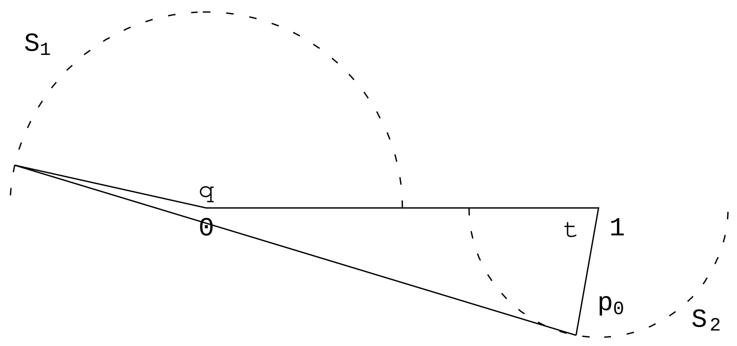

Figure 9.

Now if one reaches $\tilde{p}$ before reaching p_1, (i.e. $[\tilde{q}, \tilde{p}]$ passes through 1), then we begin moving $\tilde{q}$ along S_1 to the left and begin the same construction again. Clearly, after a finite number of steps we will meet p_1 or q_1.

So suppose we meet p_1, moving p to the right, fixing $\tilde{q}$; the segment $[\tilde{q}, p_1]$ meets I. Move $\tilde{q}$ to the left on S_1 until one reaches q_1: one can not first get to the point $\tilde{\tilde{q}}$ where the line through 0 and p_1 meets S_1, since q_1 is not on the segment of S_1 between $-||q||$ and $\tilde{\tilde{q}}$: otherwise Γ_1 would not be a figure eight; see figure 10.

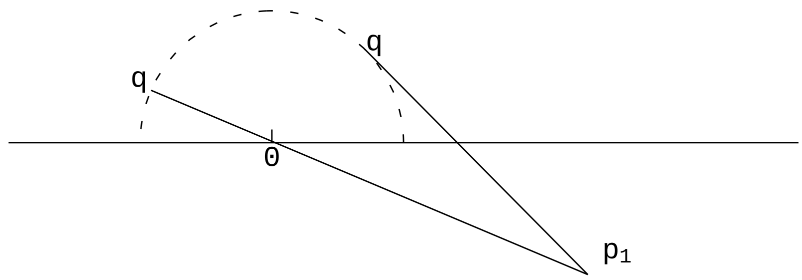

Figure 10.

This proves $||q_1 - p_1|| > ||q - p||$.

We have shown that the magnitudes of θ and τ compare well: if $\theta_1 \geq \theta$ then $\tau_1 \leq \tau$. To complete the proof of the lemma we must show the inner angles at q and p behave as we state. Let α and α_1 denote the inner angles of Γ and Γ_1 at q and q_1 . We want to show that if $\theta_1 \geq \theta$ then $\alpha_1 \leq \alpha$.

Consider figure 11. We leave the reader to verify that if both θ and α increase (decrease) then $|x - y|$ decreases (increases). The argument is similar and simpler than the previous argument where we compared θ and τ.

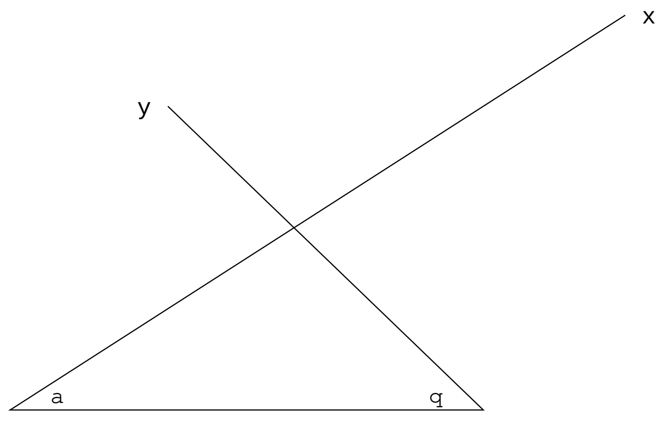

Figure 11.

This proves Cauchys' lemma for the figure eight.

REMARK 1: : For spherical figure eights, the hypothesis that the sides be of length strictly less than π is necessary. For example, let $a = (0, -1, 0),\ b = (0, 0, 1),\ c = -b,\ d = -a$. Take Γ the figure eight a, b, c, d, where the geodesic of Γ joining b to c has $(1, 0, 0)$ as tangent at b and the geodesic joining d to a has $(1, 0, 0)$ as tangent at d. Take $\Gamma' = a, b', c', d$, where b' is near b, b' on the geodesic of Γ joining b to c and $c' = -b'$.

Next we have the combinatorial lemma.

Graph lemma Let Γ be a compact graph on the sphere or projective plane and denote by $F(\Gamma)$ the number of connected components of the complement of Γ. Let s be the number of connected components of Γ. Let $\chi(\Gamma)$ be the Euler characteristic of Γ: the number of vertices minus the number of edges. Then

$$\chi(\Gamma) + F(\Gamma) = s + 1,$$

on the sphere, and

$$\chi(\Gamma) + F(\Gamma) \geq s,$$

on the projective plane.

PROOF: The spherical case was already proved by Cauchy; we prove the lemma for graphs on the projective plane P.

The reader not familiar with the topology of graphs and surfaces may consult the text of W. Massey: Algebraic Topology, An Introduction, Harcourt 1967. We make some remarks. the projective plane P can be obtained by identifying a Möbius strip with a disk by glueing them along their boundaries. Then one can see that a smooth Jordan curve C on P either bounds a disk on P or the boundary of a tubular neighbourhood of C is a connected Jordan curve C_1 and C_1 bounds a disk on P that is disjoint from C. Using this fact it is not too difficult to follow the following arguments.

First assume $s = 1$. Γ has the homotopy type of a point or a bouquet of k circles: $S(k)$. In the first case $\chi(\Gamma) = 1 \geq s$, so we can assume Γ has the homotopy type of $S(k)$. When $k = 1$, then $\chi(\Gamma) = 0$ and $F(\Gamma) \geq 1 = s$. When $k > 1$ and each cycle in Γ is null homotopic in P, then $\chi(\Gamma) + F(\Gamma) = 2$. If one of the cycles of Γ is not null homotopic then $F(\Gamma) = k$ and $\chi(\Gamma) + F(\Gamma) = (1 - k) + k = 1 = s$.

Now we proceed by induction on s. Write $\Gamma = \Gamma_1 + \Gamma_2$ where Γ_1 is a connected component of Γ which is inside a topological disk D, D disjoint from Γ_2. Applying the inductive hypothesis to Γ_2 we have

$$\chi(\Gamma_2) + F(\Gamma_2) \geq s - 1.$$

Clearly $\chi(\Gamma_1) + F(\Gamma_1) = 2$. Thus

$$\chi(\Gamma) + F(\Gamma_1) + F(\Gamma_2) \geq s + 1.$$

The lemma now follows since $F(\Gamma_1) + F(\Gamma_2) = F(\Gamma) + 1$.

Before proceeding to prove theorem 1, we will show how Cauchy could have proved the heptahedron is globally rigid, using the spherical figure eight lemma and the combinatorial lemma for the projective plane P. We hope this will help the reader to understand the ideas that follow.

The heptahedron is described in section 7, and it is also described in detail in [**6**]. Here we need to know it is a polyhedron modeled on the projective plane P, such that each vertex v is a cone on a figure eight to which the spherical figure eight lemma applies.

Let M be such a polyhedron in $\mathbf{R}^3$ and assume N is a polyhedron in $\mathbf{R}^3$ that is intrinsically isomorphic to M (so the corresponding faces of M and N are congruent). Following Cauchy, consider the link of a vertex v of M, and the link of the vertex w of N corresponding to v. These are spherical figure eights to which the spherical figure eight lemma applies. The corresponding inner angles of the figure eight are all equal or the comparison of the inner angles yields four sign changes as one traverses $lk(v)$ once. When the corresponding inner angles are all equal then $lk(v)$ is congruent to $lk(w)$ and all the inner angles along the edges of M at v are equal to the corresponding inner angles along the edges of N at w. If this happens for each vertex v of M then M and N are congruent in $\mathbf{R}^3$. To see this, do a euclidean motion to match a face F of N with its corresponding face of M. Then (after a possible symmetry of N through F) all the faces of N contiguous to F match up to the faces of M contiguous to F. Considering the remaining contiguous faces, it is clear that M and N coincide.

Now if there are two corresponding links, $lk(v)$ and $lk(w)$ which are not congruent, then we mark the edges in the star of v with a plus if the inner angle along this edge (which is the same as the inner angle at the corresponding vertex of $lk(v)$) is larger than that of $lk(w)$. Otherwise we mark it with a minus. By the spherical figure eight lemma, the total number of sign changes as one traverses the star of v once is equal to four.

Since M is modeled on the projective plane, the graph lemma, together with a counting argument which we will give in the proof of theorem one, shows it is not possible to have such a marking of the edges of M. Therefore no edge is marked and M is congruent to N.

Proof of Theorem 1: Let v be a vertex of M and $g(v)$ its Gaussian image. The lengths of the sides of $g(v)$ are given by the distance between the normals to the faces at v; each side of $g(v)$ comes from two adjacent faces of M at v. We call this side length the dihedral angle of the edge the two faces share, however it is clear that this dihedral angle depends upon the chosen orientations of the faces of M; it may not be the "small angle" between the faces. By the small (or *inner*) angle between two contiguous faces we mean the angle between 0 and π between the faces.

Now if N is sufficiently close to M, each face $\tilde{F}$ of N acquires a natural orientation from the face F of M which it is close to: $z(\tilde{F})$ is the unit normal to $\tilde{F}$ such that $\langle z(\tilde{F}), z(F)\rangle > 0$.

Next we show that if N is sufficiently close to M and if each face $\tilde{F}$ of N has the same interior angles as the face F of M which it approximates, then for each vertex $\tilde{v}$ of N, its Gaussian image $g_N(\tilde{v})$ is an embedded convex polygon on the sphere which is close to the polygon $g(v)$ (v the vertex of M which $\tilde{v}$ approaches), and whose corresponding interior angles are equal.

Clearly the vertices of $g_N(\tilde{v})$ are close to the vertices of $g(v)$, since for each face F of M, the unit normal to the face $\tilde{F}$ of N that approximates it, is close to $z(F)$, and the vertices of $g(v)$ are the points $z(F)$, F a face of M having v as vertex. So that all it remains to verify is that the interior angles of $g(v)$ and $g_N(\tilde{v})$ are the same, as each is traversed once in the same sense.

Let F_0, F_1, F_2 be adjacent faces of M at a vertex v of M, with F_1, the middle face of the three. Then the angle of $g(v)$ at $z(F_1)$ depends upon the following data: the interior angle of the face F_1 at v, the normal orientations $z(F_i)$, and whether F_0 and F_2 are going "up" or "down"with respect to F_1, along the contiguous edges $F_0 \cap F_1$, and $F_1 \cap F_2$. This dependence is made precise in lemma 3. For example, at a convex vertex with all faces oriented outward, the angle of $g(v)$ at $z(F_1)$ is $\pi - \alpha$ where α is the angle of F_1 at v. For a simple saddle vertex v, with all faces at v consistently oriented, the angle of $g(v)$ at $z(F_1)$ is α, the angle of F_1 at v. We can take ε small enough so that the "up" or "down" relations of the $\widetilde{F}_i$'s are the same as those of the F_i's. Since the orientations are the same for both we can conclude that the interior angles at $z(F_i)$ and $z(\widetilde{F}_i)$ coincide. Now it is clear that if N is sufficiently close to M and each face $\tilde{F}$ of N (close to F of M) has the same interior angles as F, then each $g_N(\tilde{v})$ has the same interior angles as $g(v)$ and is an embedded convex polygon.

The proof now proceeds along the same lines as Cauchy's. Let N be so close to M so that for each vertex v of M, $g_N(\tilde{v})$ is an embedded convex polygon, close to $g(v)$, and with the same interior angles between its sides. To each edge ℓ of M having v as vertex, we assign a plus, minus or nothing, by considering the edge of $g(v)$, coming from ℓ, and comparing the length of this edge of $g(v)$ to the length of the corresponding edge of $g_N(\tilde{v})$; mark ℓ with $+$ if the second length is greater than the first, $-$ if it is less than the first, and no marking when they are of equal length.

By the dual Cauchy lemma, the index of the vertex v (the number of sign changes as one goes around v once on M) is zero or at least four.

If the index of each vertex of M is zero, then all the dihedral angles of M and N are the same. Then take any face F of M and move N by a Euclidean motion so that $\tilde{F}$ coincides with F. Clearly this motion makes N coincide with M.

Next one proves that a contradiction is reached if any vertex has a non zero index.

Let Γ be the graph on M composed of all marked edges of M, $F(\Gamma)$ the number of connected components in the complement of Γ.

Let F_k be the number of regions (i.e., connected components of $M-\Gamma$) having k edges in its boundary. There is no region with one or two edges in its boundary so $k \geq 3$. We have

$$F(\Gamma) = F_3 + F_4 + \cdots + F_m,$$

where m is the maximal number of edges in the boundary of a region. An edge in a region is counted twice if it is traversed once in each direction when the boundary of the region is traversed once.

Let L be the number of edges in Γ.

Then

$$2L = 3F_3 + 4F_4 + \cdots + mF_m.$$

Define $T = \sum_{v\in\Gamma} \mathrm{Ind}(v)$. We know that $T \geq 4V$, V the number of vertices of Γ, by the dual Cauchy lemma.

In a region with k edges in its boundary, the total number of sign changes when one traverses the boundary of the region once, is at most k if k is even, and at most $k-1$ if k is odd. Thus

$$T \leq 2F_3 + 4F_4 + 4F_5 + 6F_6 + \ldots,$$

and since $4V \leq T$, we have

$$4V \leq 2F_3 + 4F_4 + 4F_5 + 6F_6 + \ldots. \tag{1}$$

Now the graph lemma implies that

$$V - L + F(\Gamma) \geq 1,$$

when M is the sphere or projective plane.

Hence

$$\begin{aligned}
&4V - 4 \geq 4L - 4F(\Gamma) \\
&= 2(3F_3 + 4F_4 + \cdots + mF_m) - 4(F_3 + F_4 + \cdots + F_m) \\
&= 2F_3 + 4F_4 + 6F_5 + 8F_6 + \cdots + (2m-4)F_m.
\end{aligned}$$

Subtracting the inequality 1 from this last inequality, we have

$$-4 \geq 2F_5 + 2F_6 + 4F_7 + \ldots,$$

which is impossible.

Remark 2: In the statement of theorem 1, we may relax the hypothesis M isomorphic to N and assume the interior angles of corresponding faces of M and N are equal (the lengths of corresponding sides may be distinct). Then the proof of theorem 1 still shows the dihedral angles of M and N are the same, along corresponding edges. Stoker calls this "isogonality" of M and N [**9**].

Remark 3: Theorem 1 can be extended in the following way using the spherical figure eight lemma. For the given face orientations of M do not demand that the figure eight vertices be simple; however, require that for each edge ℓ of M, joining a simple vertex to a cone on a figure eight vertex, the dihedral angle of ℓ for the simple vertex is the small angle of the figure eight (of the other vertex of ℓ). With these conditions the proof of theorem 1 and the figure eight lemma allows us to mark each edge of M with a plus or minus (or nothing) by comparing the inner angles between two faces along an edge, so that the index of each vertex of M is zero or four (greater than or equal to four if the edge joins two simple vertices) showing that M is locally rigid.

Remark 4: The notion of herisson was introduced to study certain (perhaps singular) surfaces in $\mathbf{R}^3$ which have smooth bijective Gauss maps. Ovaloids are smooth examples of herissons and in general, any smooth function h on S defines a herisson with h as support function. Minimal surfaces, and algebraic singularities can also be herissons, [**7**], [**8**].

An example of a herisson of revolution, not convex, is indicated in figure 12. There are two positively curved parts, topologically disks, and a negatively curved annulus. Notice that the orientation that makes the Gauss map bijective does not orient the surface globally.

Now we define a *polyhedral herisson* as a polyhedron M in $\mathbf{R}^3$, admitting a face orientation, that makes the Gauss map of M a bijection. This means that $g(v)$ is embedded for each vertex v of M and the domains bounded by the $g(v)$'s cover

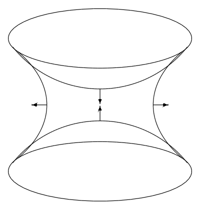

Figure 12.

bijectively (i.e., they partition) the sphere. An example of a polyhedral herisson having the same "structure" as the example of figure 12, is given in figure 13.

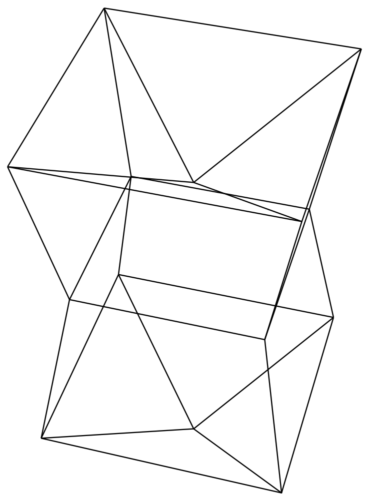

Figure 13.

Notice, also that the house with the roof caved in of figure 2, is a polyhedral herisson. The sunken roof and floor are oriented into the house and the walls are oriented out.

It is natural to ask what properties of convex spaces also hold for herissons, (rigidity, Minkowski, etc.).

4. Properties of the Gauss map.

In our proof of theorem 1 we used the fact that the angles between the sides of $g(v)$, v a vertex of M, can be calculated explicitly knowing the local geometry of M near v, and in particular, they vary continuously when one deforms M, providing the dihedral angles are strictly between 0 and π. In this section we will make this explicit and establish some results that will permit us to obtain a global rigidity theorem.

For v a vertex of M, we call the interior angle between two sides of $g(v)$, the angle between 0 and π between the two sides. A coherent orientation of $st(v)$ will denote a choice $F_1, F_2, \ldots, F_m$, of an orientation of $st(v)$: one turns around the faces F_i at v in the order $F_1, F_2, \ldots, F_m, F_1$. Thinking of F_i in $\mathbf{R}^3$, a coherent orientation $\hat{z}$ of $st(v)$, induces an orientation $\hat{z}(F_i)$ of each face.

Now suppose we have chosen a face orientation $z(F_i)$, of each of the faces F_i at v, and a coherent orientation $\hat{z} = [F_1, F_2, \ldots, F_m]$ of $st(v)$. For each face F_i we define $\sigma(F_i) = 1$ if $z(F_i)$ is the orientation of F_i induced by $\hat{z}$; otherwise $\sigma(F_i) = -1$. Clearly changing the coherent orientation changes the sign of σ.

Before discussing our local analysis of the geometry of the Gauss map, we give another local rigidity result. Define a face orientation at a vertex v to be alternating if the sign of $\sigma(F_i)$ alternates with respect to a coherent orientation $\hat{z} = [F_1, \ldots, F_m]$ of $st(v)$. Clearly, this does not depend on which coherent orientation of $st(v)$ we choose.

Theorem 2. *Let M be a polyhedron modeled on the sphere or projective plane. Suppose that one can choose for each vertex v a face orientation of $st(v)$ that is coherent or alternating. Then if all $g(v)$ are convex and embedded (for the chosen* *local face orientations), M is ε-rigid.*

This theorem applies to the heptahedron: at each vertex, choosing the face orientations $\{+,-,+,-\}$ makes the Gauss image convex embedded and each vertex is alternating. There is no face orientation of this example which makes each vertex simple.

Also theorem 2 applies to any sphere or projective plane with convex, simple saddle, and figure eight vertices.

PROOF OF THEOREM 2: Consider a vertex v of M and and edge ℓ of M having v as a vertex. Let F_1, F_2 be the faces of M having ℓ as an edge. Denote by $z_v(F_1)$, $z_v(F_2)$ the local face orientations of F_1, F_2 which are given, making v coherent or alternating. Let $d(a,b)$ denote the distance on the unit two-sphere between two unit vectors a and b.

Now when N is a small enough perturbation of M we can give N the local face orientations coming from those of M. Let $\tilde{v}$, $\tilde{\ell}$, $\widetilde{F}$ denote the vertex, edge, face of N close to v, ℓ, F of M. Clearly if v is coherent (alternating) then $\tilde{v}$ is coherent (alternating); $z_{\tilde{v}}(\widetilde{F}_1)$ and $z_{\tilde{v}}(\widetilde{F}_2)$ will orient (not orient) $\widetilde{F}_1+\widetilde{F}_2$ if $z_v(F_1)$ and $z_v(F_2)$ orient (not orient) F_1+F_2.

We mark the side ℓ of M as follows. If v is coherently oriented then assign plus (minus) to ℓ if

$$d(z_v(F_1),z_v(F_2)) > \ (<)\ d(z_{\tilde{v}}(\widetilde{F}_1),z_{\tilde{v}}(\widetilde{F}_2)).$$

If v is alternating then assign plus (minus) to ℓ if

$$d(z_v(F_1),z_v(F_2)) < (>)\ d(z_{\tilde{v}}(\widetilde{F}_1),z_{\tilde{v}}(\widetilde{F}_2)).$$

The inner angle between the faces F_1, F_2 is given by $d(z_v(F_1),z_v(F_2))$ if $z_v(F_1)$ and $z_v(F_2)$ do not orient F_1+F_2, and it is $\pi - d(z_v(F_1),z_v(F_2))$ otherwise.

Suppose v and w are the vertices of an edge ℓ. When both v and w are coherently oriented, then the markings are as were done in the proof of Theorem 1 and one can apply Cauchy's Dual Lemma. By the definition of the markings in case both are alternating we get that the markings correspond to the opposite of the ones done in the coherent case; however we can still use Cauchy's Dual Lemma since what matters are the sign changes. Finally, in the more generaral case that v is coherent and w alternating, the marking ℓ receives from v is the same marking ℓ receives from w. It is plus (minus) if the inner angle between F_1 , F_2 along ℓ in M is smaller (larger) than the inner angle between $\widetilde{F}_1$, $\widetilde{F}_2$ along $\tilde{\ell}$ in N.

At a vertex v of M, if one edge is marked, then the total number of sign changes as one goes once around $st(v)$ is at least four. Now the proof given in theorem 1 shows no edge of M is marked and N is congruent to M. ▮

5. The geometry of the Gauss map.

In this section we will make precise the relationships between the inner angles of $g(v)$ and the intrinsic geometry of M at v together with the local face orientations at v.

Let $\hat{z} = [F_1, \dots, F_m]$ be a coherent orientation of $st(v)$. We define $\gamma(F_{i+1})$ to be 1 if F_{i+1} is in the half space of $\mathbf{R}^3$, towards which $\hat{z}(F_i)$ is pointing (i.e. F_{i+1} is going up, w.r.t. $\hat{z}(F_i)$, along $F_i \cap F_{i+1}$). Otherwise define $\gamma(F_{i+1}) = -1$.

Observe that $\gamma(F_i)\gamma(F_{i+1}) = \pm 1$ depending on whether or not the faces $f(F_{i-1})$ and $f(F_{i+1})$ are on the same side of the plane containing $f(F_i)$. If this index is positive we call the face *convex*; otherwise we call it alternating.

Once a coherent orientation of $st(v)$ is chosen, we define the notion of left and right turns at a vertex $z(F_i)$ of $g(v)$ as follows. The coherent orientation of $st(v)$ naturally orients the curve $g(v)$ on the sphere. Then we say the inner angle of $g(v)$ at the vertex $z(F_i)$ of $g(v)$, β_i, is a left (right) turn angle if $(z(F_i) - z(F_{i-1})) \times$

$(z(F_{i+1}) - z(F_i))$ points outside (inside) the sphere S (i.e., as one traverses $z(F_i)$, along $g(v)$, one turns left, for the standard orientation of S). Notice that changing the coherent orientation of $st(v)$, changes left to right.

In the following lemma 3 we make no assumption on the vertex v nor on the type of its Gaussian image.

LEMMA 3. *Let $g(v)$ be the Gauss map of a vertex v with respect to a face orientation z. Suppose that for a coherent orientation $\hat{z}$ of $st(v)$ the faces are ordered as $\{F_1, F_2, \ldots, F_n\}$. Let α_i be the interior angle of F_i at v and β_i the interior angle of $g(v)$ at $z(F_i)$. Then*

(1) *$\beta_i = \pi - \alpha_i$ or $\beta_i = \alpha_i$ depending on whether*

$$\sigma(F_{i-1})\sigma(F_{i+1})\gamma(F_i)\gamma(F_{i+1}) = 1,$$

or -1. (In particular, β_i depends continuously on α_i when one deforms the polyhedron under consideration).

(2) *the angle at $z(F_i)$ turns left or right depending on whether*

$$\sigma(F_{i-1})\sigma(F_i)\sigma(F_{i+1})\gamma(F_i)\gamma(F_{i+1}) = 1,$$

or -1

PROOF: Suppose that the vertices in F_i adjacent to v are b and c. Let a be the point in the segment $[vb]$ that makes $[ac]$ perpendicular to $[vb]$ at a. Similarly let d be the point in $[vc]$ that makes $[db]$ perpendicular to $[vc]$; let o be the intersection point of $[ac]$ and $[bd]$; see figure 14. Consider the face $f(F_i)$ to be centered at o the zero of the plane $x_3 = 0$. Assume that for the coherent orientation $\tilde{z}$ of v, $\tilde{z}(F_i) = (0, 0, 1)$, with faces ordered $\{F_1, F_2, \ldots, F_n\}$. Thus, the edge $[vb]$ is common to F_{i-1} while $[vc]$ is common to F_{i+1}. Thus, $z(F_{i-1})$ must be contained in the plane

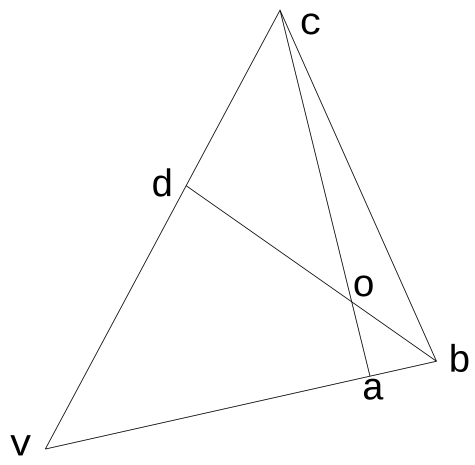

Figure 14. Basic Lemma

P_1 which is perpendicular to $x_3 = 0$ and to the edge $[vb]$, while $z(F_{i+1})$ must be in the plane P_2 which is perpendicular to $x_3 = 0$ and $[vc]$. In figure 14 we see that $[ac] \subset P_1$ and $[db] \subset P_2$. The angle at $z(F_i)$ will be equal to one of the following angles $\angle aob$, $\angle aod$, $\angle cob$, $\angle cod$, since $z(F_{i-1})$ must belong to the circle $S \cap P_1$ while $z(F_{i+1})$ must belong to the circle $S \cap P_2$. Which of these angles it is depends on the face orientation of the adjacent faces and the positions of their images by f with respect to the plane containing $f(F_i)$. Recall that $\gamma(F_{i+1}) = 1$ if $f(F_{i+1})$ is in the half-space $x_3 > 0$, and equals -1, otherwise.

We can easily check that the angle starts at a (i.e. $z(F_{i-1})$ is in the half-sphere perpendicular to $[ca]$ and containing a) if

$$\sigma(F_{i-1}) = 1 \quad \text{and} \quad \gamma(F_i) = -1$$
$$\sigma(F_{i-1}) = -1 \quad \text{and} \quad \gamma(F_i) = 1.$$

Similarly, the angle starts at c if

$$\sigma(F_{i-1}) = 1 \quad \text{and} \quad \gamma(F_i) = 1$$
$$\sigma(F_{i-1}) = -1 \quad \text{and} \quad \gamma(F_i) = -1$$

On the other hand the angle ends at b if

$$\sigma(F_{i+1}) = 1 \quad \text{and} \quad \gamma(F_{i+1}) = 1$$

$$\sigma(F_{i+1}) = -1 \quad \text{and} \quad \gamma(F_{i+1}) = -1$$

Similarly the angle ends at d if

$$\sigma(F_{i+1}) = 1 \quad \text{and} \quad \gamma(F_{i+1}) = -1$$

$$\sigma(F_{i+1}) = -1 \quad \text{and} \quad \gamma(F_{i+1}) = 1.$$

Since $[ca] \perp [vb]$ and $[bd] \perp [vc]$, we have that

$$\angle aod = \angle cob = \pi - \alpha_i$$

$$\angle aob = \angle cod = \alpha_i.$$

Thus in order to prove 1) it is sufficient to check that

$$\sigma(F_{i-1})\sigma(F_{i+1})\gamma(F_i)\gamma(F_{i+1}) = 1, \tag{a}$$

for $\angle aod$ and $\angle cob$, and that

$$\sigma(F_{i-1})\sigma(F_{i+1})\gamma(F_i)\gamma(F_{i+1}) = -1, \tag{b}$$

for $\angle aob$ and $\angle cod$. To check it for $\angle aod$, we see from the above statement that if the angles start at a then $\sigma(F_{i-1})\gamma(F_i) = -1$ while if it ends at d then $\sigma(F_{i+1})\gamma(F_{i+1}) = -1$, so that their product gives $+1$, as we wanted to show. The other angles are checked similarly, proving item 1).

In order to prove 2) we must consider $\sigma(F_i)$. Suppose first that $\sigma(F_i) = 1$, i.e. that $z(F_i) = (0, 0, 1)$. Then clearly $\angle aod$ and $\angle cob$ are left turns in the plane $x_3 = 0$ and hence in S, while $\angle aob$ and $\angle cod$ are right turns. Since $\sigma(F_i) = 1$, the product in 2) equals the product in 1). Now by formula (a) in the proof of 1) we conclude that 2) holds in the case $\sigma(F_i) = 1$.

On the other hand, if $\sigma(F_i) = -1$, then the angles that are right turns in the plane $x_3 = 0$ are left turns on the sphere, showing that $\angle aob$ is a left turn. Then exactly as in the last paragraph, using formula (b) we see that 2) holds when $\sigma(F_i) = -1$. ∎

We will need the following facts about fold and figure-8-cone vertices.

To simplify the notation we will assign a $+$ to a face that has the same oriention as that given by a coherent orientation of the star of the vertex in question. For a fold vertex we choose a coherent orientation so that the orientations of the faces on the convex hull, F_1 and F_2 are $+,+$, while the orientation of F_3 and F_4 are $-,-$. With this choice the relation between the inner angles of the faces at v, α_i, and the interior angles of the Gauss image polygon, β_i, are $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, $\beta_3 = \pi - \alpha_3$, $\beta_4 = \pi - \alpha_4$. By intrinsic metric arguements, we obtain the following.

Lemma 4. *For a fold,*

$$\alpha_1 + \alpha_2 > \alpha_3 + \alpha_4,$$

where α_1 and α_2 are the interior angles of the faces on the convex hull.

Proof: Using Gauss-Bonnet on the quadrilateral given by the Gauss map, we obtain

$$\sum_{i=1}^{4}(\pi - \beta_i) + A = 2\pi,$$

where A is the spherical area of the convex domain bounded by $g(v)$. But on a fold where the orientations of the faces on the convex hull, F_1 and F_2 are $+,+$, the relation between the inner angles of the faces at v, α_i, and the interior angles of the Gauss polygon, β_i, are $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, $\beta_3 = \pi - \alpha_3$, $\beta_4 = \pi - \alpha_4$. Thus, the above equation can be written as

$$\alpha_3 + \alpha_4 - \alpha_1 - \alpha_2 = -A \leq 0,$$

proving the inequality.

LEMMA 5. *For a figure 8,*

$$\alpha_1 + \alpha_3 < \alpha_2 + \alpha_4,$$

where α_1 and α_3 are the two faces on the boundary of the convex hull of the cone on the figure 8.

The proof of this lemma follows easily from the triangle inequality applied to the two triangles forming the figure 8; we leave the details to the reader.

6. A global rigidity theorem.

In this section we call a vertex *simple* if it is either convex, or has four or less edges coming into it, and its Gaussian image is a convex embedded curve.

THEOREM 6. *Let M be a polyhedron in $\mathbf{R}^3$ modeled on the sphere or projective plane and assume all vertices of M are simple for some face orientation of M. Then it is globally rigid in the class of polyhedra isomorphic to M whose vertices are also simple for the same face orientation as that of M.*

PROOF OF THEOREM 6: Let M and N be isomorphic polyhedra satisfying the hypothesis of theorem 6. Consider a vertex v of M. First we will show that Cauchy-Steinitz can be applied to $g_M(v)$ and $g_N(v)$ (v the corresponding vertex of N in the expression $g_N(v)$). By this we mean that the interior angles between the edges of $g_M(v)$ and $g_N(v)$ (the β_i) will be shown to be equal. Then we will show that the edges of M can be unambiguously marked in terms of the differences of the side lengths of $g_M(v)$ and $g_N(v)$. Finally, we will prove that when there are no edges of M marked then M is congruent to N.

Let v be a vertex of M. We consider first the case when σ is a constant sign at $st(v)$. Choose a coherent orientation of $st(v)$ that makes $\sigma = 1$ at each face of $st(v)$. Since $g(v)$ is a convex embedded polygon, one always turns the same way at each angle of $g(v)$ (always left or always right). Then lemma 3 part 2) shows that $\gamma(F_i)\gamma(F_{i+1}) \equiv 1$, or -1, since we are assuming $\sigma \equiv 1$. In the case $\gamma(F_i)\gamma(F_{i+1}) \equiv 1$, lemma 3 part 1) implies that for all i, $\beta_i = \pi - \alpha_i$. Now Gauss-Bonnet applied to the convex spherical domain D determined by $g(v)$ gives us that

$$\sum_i^n \pi - \beta_i + \int_D K = 2\pi,$$

or

$$\sum_i^n \alpha_i + A = 2\pi,$$

where A is the spherical area of D. It is natural to define the *curvature at a vertex* at a vertex v, K_v, as $2\pi - \sum \alpha_i$; this curvature is intrinsic, i.e., it is the same for isomorphic polyhedra. With this definition one obtains a Gauss-Bonnet theorem for polyhedral surfaces (see [**1**]). Thus the above formula gives us that $K_v = 2\pi - \sum \alpha_i = A > 0$.

In the case $\gamma(F_i)\gamma(F_{i+1}) \equiv -1$, lemma 3 implies that $\beta_i = \alpha_i$, and Gauss-Bonnet implies that

$$\sum_i^n \pi - \alpha_i + A = 2\pi,$$

from which we conclude that $K_v = 2\pi - \sum_i^n \alpha_i = \pi(2 - (n-2)) - A \leq -A < 0$.

In conclusion, if a simple vertex has a coherent face orientation then depending on whether the sign of the curvature is positive or negative we have, for all i, $\beta_i = \pi - \alpha_i$ or $\beta_i = \alpha_i$. Since the sign of the curvature of a vertex is intrinsic and the property of its star having a coherent orientation is satisfied simultaneously in M and N, we get that for these types of vertices we can apply the Cauchy-Steinitz lemma.

We will now prove the following fact: *If v is a convex vertex of M with at least four faces in its star and if v is simple for some local face orientation at v then the local orientation is coherent.* Let us fix a coherent orientation for $st(v)$ and let γ denote its gaussian image. Let Γ denote the gaussian image of v for the local face orientation which makes v simple. We know that $\gamma_i\gamma_{i+1} \equiv 1$ since v is a convex vertex so by part 2) of lemma 3, we conclude that $\sigma_i = \sigma_{i+3}$ for all i. If the local face orientation is not coherent, then we can assume the σ's of F_1, F_2, F_3, F_4 are $+, +, -, +$. Let $z_i = z(F_i)$ denote the face orientation of F_i, which makes v simple (not the fixed coherent orientation of $st(v)$ that we chose).

We can suppose z_1, z_2 are on the equator of S and (since Γ is convex) that z_3, z_4 are in the northern hemisphere. Now γ is also convex and $z_1, z_2, -z_3, z_4$ belong to γ. This is impossible since $-z_3$ is in the southern hemisphere, so γ would not be convex.

Now we know that if a simple vertex v of M is not coherent and has at least four faces in its star, then v has exactly four faces in its star and v is a fold or cone on a figure eight vertex.

When there are exactly three faces in $st(v)$, then $g(v)$ is a geodesic triangle hence it is determined by its inner angles. So we need not discuss markings for the edges of this vertex. So we now assume there are at least four faces in $st(v)$.

Next we will show that Cauchy-Steinitz applies to fold and figure eight vertices; in particular a fold (or figure eight) vertex of M will also be a fold (figure eight) vertex of N.

Let v be a fold or figure eight vertex of M. Orient $st(v)$ by $\{F_1, F_2, F_3, F_4\}$. Since the σ_i must change sign we can assume that the face orientation of $\{F_1, F_2, F_3, F_4\}$ is $\{+, \pm, \pm, -\}$.

Suppose first that $\sigma_2 = +1$. Then the equations of Lemma 3, 2), gives us that

$$\sigma_3\gamma_2\gamma_3 = \sigma_3\sigma_4\gamma_3\gamma_4 = \sigma_3\sigma_4\gamma_4\gamma_1 = \sigma_4\gamma_1\gamma_2$$

This implies that $\sigma_3 = \sigma_4$. Thus, $\sigma = \{+,+,-,-\}$. Observe then that the index $a_i = \sigma_{i-1}\sigma_{i+1}\gamma_i\gamma_{i+1}$ that it is used in Lemma 3, 1) to determine whether $\beta_i = \pi - \alpha_i$ or $\beta_i = \alpha_i$ satisfies $a_i = c\sigma_i$ where $c = \pm 1$ depending on whether $g(v)$ turns right or left. Consider now the case where the $g(v)$ turns right. Then, we have in this case that $a_1 = a_2 = -1$ and $a_3 = a_4 = 1$. This implies that $\gamma_1\gamma_2 = \gamma_2\gamma_3 = 1$ which in turn occurs if and only if F_1 and F_2 are the convex faces of a fold. However, by Lemma 4, the fact that F_1 and F_2 are the convex faces is intrinsic; that is, the corresponding faces $\tilde{F}_1$ and $\tilde{F}_2$ of N are the convex faces, giving that $\tilde{\gamma}_1\tilde{\gamma}_2 = \tilde{\gamma}_2\tilde{\gamma}_3 = 1$, $\tilde{a}_1 = \tilde{a}_2 = -1$ and $\tilde{a}_3 = \tilde{a}_4 = 1$. Thus the inner angles satisfy $\tilde{\beta}_i = \beta_i$, and, since the σ_i's are the same for both, $g_N(v)$ also turns right. In the case that $g(v)$ turns left a similar argument shows F_3 and F_4 are the convex faces and one can apply Cauchy-Steinitz in the same way.

Finally, we consider the case where $\sigma = \{+,-,\pm,-\}$. Since $\sigma_2 = -1$ the equations of Lemma 3, 2), gives us that

$$-\sigma_3\gamma_2\gamma_3 = \sigma_3\gamma_3\gamma_4 = -\sigma_3\gamma_4\gamma_1 = \gamma_1\gamma_2$$

This implies that $\sigma_3 = 1$. Thus, $\sigma = \{+,-,+,-\}$. Suppose first that the Gauss map turns left at v. As in the proof above of the fold case, we have that the indices a_i satisfy $a_1 = a_3 = 1$ and $a_2 = a_4 = -1$, which imply that $\gamma_1\gamma_2 = \gamma_3\gamma_4 = 1$ and $\gamma_2\gamma_3 = \gamma_4\gamma_1 = -1$. This is equivalent to say that F_1 and F_3 are the convex faces while F_2 and F_4 are the alternating faces. Finally, we point out that by Lemma 5, the intrinsic metric determines the sides that are on the convex hull. that is, the corresponding faces $\tilde{F}_1$ and $\tilde{F}_3$ of N are the convex faces, giving that $\tilde{\gamma}_1\tilde{\gamma}_2 = \tilde{\gamma}_3\tilde{\gamma}_4 = 1$, in turn implying that $\tilde{a}_1 = \tilde{a}_3 = 1$ and $\tilde{a}_2 = \tilde{a}_4 = -1$. Thus the inner angles satisfy $\tilde{\beta}_i = \beta_i$, and, since the σ_i's are the same for both, $g_N(v)$ also turns left. In the case that $g(v)$ turns right a similar argument shows F_2 and F_4 are the convex faces and one can apply Cauchy-Steinitz in the same way.

Now suppose M and N are parametrized by f_1 and f_2 respectively. We have seen that Cauchy-Steinitz can be applied to each vertex v of Σ and as in the proof of theorem 1 we conclude that the dihedral angles of M are the same as those of N. Notice also that the type of the vertex v is the same for M and N: our above characterization of convex and saddle points (when the σ_i are of the same sign) and of fold and figure eight points (in terms of $\{+,+,-,-\}$ or $\{+,-,+,-\}$) proves this.

Let v be a vertex of Σ and F_1 a face at v. Perform an isometry of $\mathbf{R}^3$ so that $f_1(F_1) = f_2(F_1)$ and they have the same normal z_1 . Let F_2 be an adjacent face of F_1. Since the dihedral angle along $F_1 \cap F_2$ is the same for M and N we can assume that $f_1(F_1 \cup F_2) = f_2(F_1 \cup F_2)$ (after a possible reflection through the face $f_1(F_1)$; even if no edge is marked, M and N might still differ by an orientation-reversing space isometry. With the notations of the proof, this shows up as the possibility that $\gamma_i\tilde{\gamma}_i = -1$ for all corresponding stars of M and N. Composing f_2 say, with any orientation reversing isometry changes all $\tilde{\gamma}_i$ to their opposite). Now since v is a simple vertex of M and N of the same type, it is clear that $f_1(st(v)) = f_2(st(v))$. Let w be a vertex in the boundary of $st(v)$. There are two faces of $st(w)$ in $st(v)$ so, in fact, $f_1(st(w)) = f_2(st(w))$. Thus, continuing this reasoning, we conclude $f_1 = f_2$. ∎

REMARK 5: Suppose M is a polyhedron modeled on the sphere or projective plane and each vertex of M is a cone on a figure eight or a convex vertex. Then M is rigid among isomorphic polyhedra with the same type of vertices. The figure eight lemma and Cauchy's geometric lemma applied to the link of each vertex v and the proof of theorem 6 allows us to mark each edge of M with a plus or minus (or nothing) by comparing the inner angles between two faces along an edge, so that the index of each vertex of M is zero or four. Then the graph lemma yields all the indices are zero and M is congruent to N. For example, the heptahedron is

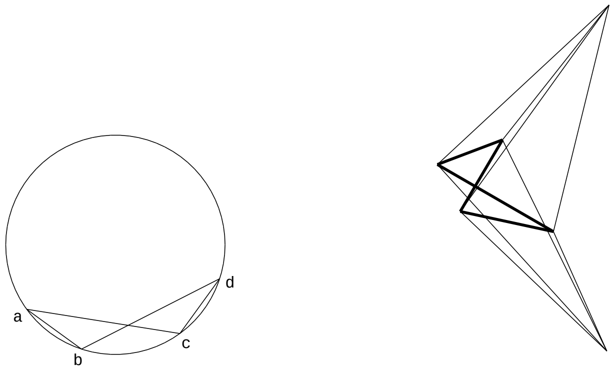

Figure 15. Bricards example

rigid. Notice that we cannot directly apply theorem 6 to this case because we do not necessarily have global face orientations that make every vertex simple.

7. Examples.

We begin with Bricard's second flexible octahedron. Consider the figure eight inscribed in a circle, with vertices a, b, c, d, chosen so that the length of the segment $[a, b]$ equals that of $[c, d]$. Moreover, choose the vertices to all lie in a semicircle, cf. figure 15.

Imagine the circle contained in the (x, y) plane and centered at $(0, 0)$. Let $p = (0, 0, 1)$ and $q = -p$. Let M be the suspension of the figure eight with vertices p and q, cf. figure 15. M is modeled on the sphere and it is flexible. The reader should convince himself of this by twisting the figure eight out of the (x, y) plane; figure 16.

Notice that p, and q are vertices of M that are cones on a figure eight. The vertices b and c are convex and a and b are fold vertices. There is no face orientation of M which makes the vertices simple.

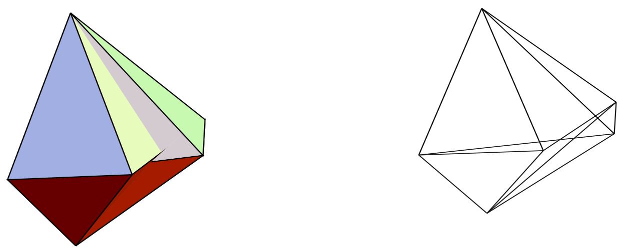

Figure 16. Bricards twisted example

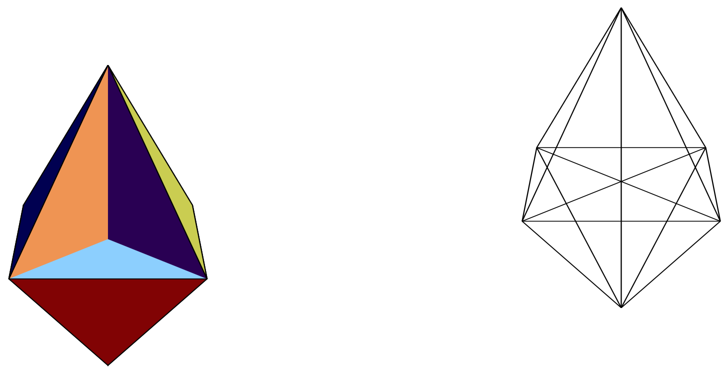

Figure 17. Heptahedron

We can make a (large) deformation of M to obtain a rigid polyhedron. Slide the vertices a and b on the circle so that a diagonal of the circle separates a, d from b and c. Then the suspension of this figure eight from p and q is rigid. Since the vertices a, b, c, d are all convex so remark 5 applies.

Here is the simplest example to which our theory applies. Let a, b, c, d be the vertices of the simple saddle vertex v of figure 4; $st(v)$ has four triangular faces. Let M be the spherical polyhedron obtained by adding two triangles to $st(v)$: the triangles b, c, d and a, c, d. Notice that one can orient the faces so that v is a simple saddle and both c and d are simple fold vertices. The vertices a and b are simple convex vertices and since there are exactly three faces in the stars of both a and b, their Gauss images are rigid for any face orientations at a and b. Theorem 1 yields rigidity of this polyhedron.

For the readers convenience, we describe the heptahedron. Begin with a rectangle inscribed in the circle with vertices a, b, c, d and let N be the octahedron which is the suspension of the rectangle from p and q. Remove from N four faces: two faces on the top, the triangles p, a, b and p, d, c, and two faces on the bottom, the triangles q, d, a and q, d, c. See figure 17.

There are four faces left. Add three faces to obtain a complex with seven faces, the heptahedron: the three faces one adds are the horizontal rectangle $[a, b, c, d]$, and the two vertical quadrilaterals over the two diagonals of the rectangle $[a, b, c, d]$; i.e., the quadrilateral p, b, q, d and the quadrilateral p, a, q, c.

Each vertex is a cone on a figure eight so M is rigid by remark 5. Notice that there is no face orientation of M that makes the vertices simple.

Bibliography

1. Banchoff, T., *Critical points and curvature for embedded polyhedra*, J. Differential Geometry **1** (1967), 245–256.
2. Connelly, R., *A counterexample to the rigidity conjecture for polyhedra*, Inst. Hautes Etudes Sci. Publ. Math. **47** (1977), 333–338.

3. Connelly, R., *"Rigidity"*, Handbook of Convex Geometry (1993), 223–271, Gruber and Wills.
4. Gluck, H., *"Almost all simply connected closed surfaces are rigid"*, Geometric Topology, Lect. Notes in Maths. **438 (Springer)** (1975), 225–239.
5. Ivanova-Karatopraklieva and Sabitov, I. Kh., *"Bending of surfaces, Part II"*, Journ. of Math. Sc. **74, No. 3** (1995), 997–1043.
6. Hilbert, D. and Cohn-Vossen, S., Chelsea (1952), 302–304.
7. Langevin, R., Levitt, G., Rosenberg, H., *"Hérrisons et multiherissons (enveloppes parametrées par leur application de Gauss""*, Singularities, Banach Center pub 20, PWN, Polish Sc. Warsaw (1988), 245–253.
8. Rosenberg, H., and Toubiana, E., *"Complete minimal surfaces and minimal herissons"*, J. Differential Geometry **27** (1988), 115–132.
9. Stoker, J.J., *"Geometrical problems concerning polyhdedra in the large"*, Comm. Pure and Appl. Maths. **21** (1968), 119–174.

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