No-slip solvability constraints revisited

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10 Feb 1994

1 Introduction

After performining a Fourier expansion (with Fourier index n) in the x-direction of our standard periodic channel of length L_x with no-slip walls located at $y = \pm 1$, the Poisson equation reads

$$\frac{d^2\psi_n}{dy^2} - \Big(\frac{2\pi n}{L_x}\Big)^2 \psi_n = -\omega_n(y)\,, \tag{1}$$

with the Dirichlet boundary conditions

$$\psi_n(-1) = \psi_n(1) = 0\,, \tag{2}$$

and the Neumann boundary conditions

$$\psi_n'(-1) = \psi_n'(1) = 0\,. \tag{3}$$

¿From now on, we will assume a fixed value of $n \neq 0$ and drop the subindex n. We also define

$$\lambda = \frac{2\pi n}{L_x}\,. \tag{4}$$

Introducing the Dirichlet Green's function $g(y|s)$, defined by

$$g_{yy} - \lambda^2 g = \delta(y-s)\,, \tag{5}$$

with the boundary conditions

$$g(-1) = g(1) = 0\,, \tag{6}$$

we can formally write the solution to the Dirichlet Poisson equation as

$$\psi_{\rm Dir}(y) = -\int_{-1}^{1} \omega(s)\, g(y|s)\, ds\,. \tag{7}$$

The solvability constraint then arises by requiring that the function ψ not only satisfies (7) but also the Neumann boundary conditions, which means that we must impose the integral constraint

$$\int_{-1}^{1} \omega(s) \Big(\frac{\partial\, g(y|s)}{\partial y}\Big)_{y=\pm 1} ds = 0\,. \tag{8}$$

In the coming sections we will describe how eq. (8) can be computed and expressed in a closed form by various methods.

1.1 Calculation of $g(y|s)$

We can immediately split the Green's function into the two parts

$$g(y|s) = \begin{cases} g^-(y) = A^- e^{\lambda y} + B^- e^{-\lambda y} & \text{for } y < s \\ g^+(y) = A^+ e^{\lambda y} + B^+ e^{-\lambda y} & \text{for } y > s. \end{cases} \tag{9}$$

Employing the outer boundary conditions

$$\begin{aligned} g^-(-1) &= 0 \\ g^+(1) &= 0\,, \end{aligned} \tag{10}$$

we find

$$\begin{aligned} g^-(y) &= A\sinh(\lambda(y+1)) \\ g^+(y) &= B\sinh(\lambda(y-1))\,. \end{aligned} \tag{11}$$

At $y = s$ we must require that $g(y)$ is continuous and that the jump in $g_y(y)$ is consistent with eq. (5), so

$$\begin{aligned} g^-(s) &= g^+(s) \\ g_y^+(s) - g_y^-(s) &= 1\,, \end{aligned} \tag{12}$$

Imposing these two inner boundary conditions, we can determine the constants A and B in eq. (11), which gives

$$g(y|s) = \begin{cases} \sinh[\lambda(s-1)]\,\sinh[\lambda(y+1)]\,/\,[\lambda\sinh(2\lambda)] & \text{for } -1 \le y \le s \\ \sinh[\lambda(s+1)]\,\sinh[\lambda(y-1)]\,/\,[\lambda\sinh(2\lambda)] & \text{for } s \le y \le 1. \end{cases} \tag{13}$$

¿From this expression, we easily can find

$$\frac{\partial g(y|s)}{\partial y} = \begin{cases} \sinh[\lambda(s-1)]\,\cosh[\lambda(y+1)]\,/\,\sinh(2\lambda) & \text{for } -1 \le y \le s \\ \sinh[\lambda(s+1)]\,\cosh[\lambda(y-1)]\,/\,\sinh(2\lambda) & \text{for } s \le y \le 1. \end{cases} \tag{14}$$

Inserting this expression into (8) and dropping the constant factors we find the Green's function solvability constraints

$$\int_{-1}^{1} \omega(s)\,\sinh(\lambda(s \pm 1))\,ds = 0\,. \tag{15}$$

Since

$$\sinh(\lambda(s \pm 1)) = \sinh(\lambda s)\cosh(\lambda) \pm \cosh(\lambda s)\sinh(\lambda)\,, \tag{16}$$

we can rewrite eq. (15) as

$$\int_{-1}^{1} \omega(s)\,\sinh(\lambda s)\,ds \pm \tanh(\lambda)\int_{-1}^{1} \omega(s)\,\cosh(\lambda s)\,ds = 0\,. \tag{17}$$

The function $\omega(s)$ will eventually be expanded in terms of Chebyshev polynomials

$$\omega(s) \approx \sum_{m=0}^{M} \omega_m T_m(s)\,, \tag{18}$$

so we will formulate the solvability constraints (17) in terms of weighted sums of the Chebyshev coefficients

$$\sum_{m=0}^{M} \omega_m b_m^{\pm} = 0\,. \tag{19}$$

These conditions are equivalent to requiring that the vector with elements ω_m is orthorgonal to both of the vectors b_m^+ and b_m^-. In the following sections we will describe different procedures for the calculation of the coefficients $b_m^{\pm}$.

1.2 Determination of $b_m^{\pm}$ by numerical Chebyshev expansions

By insertion of the Chebyshev expansion of $\omega(s)$ from (18) into the integral solvability equation (17), we find

$$\begin{aligned} &\int_{-1}^{1} \sum_{m=0}^{M} \omega_m T_m(s)\, \sinh(\lambda s)\, \sqrt{1-s^2}\, \frac{ds}{\sqrt{1-s^2}} \pm \\ \tanh(\lambda) &\int_{-1}^{1} \sum_{m=0}^{M} \omega_m T_m(s)\, \cosh(\lambda s)\, \sqrt{1-s^2}\, \frac{ds}{\sqrt{1-s^2}} \quad = \quad 0\,. \end{aligned} \tag{20}$$

We now introduce the two functions

$$\begin{aligned} C(s) &= \Lambda \cosh(\lambda s)\, \sqrt{1-s^2} \\ S(s) &= \Lambda \sinh(\lambda s)\, \sqrt{1-s^2}\,, \end{aligned} \tag{21}$$

which have been normalized by the constant

$$\Lambda \equiv \lambda e^{-\lambda}\,. \tag{22}$$

These functions have the Chebyshev expansions

$$\begin{aligned} C(s) &\approx \sum_{p=0}^{P} C_p T_p(s) \\ S(s) &\approx \sum_{p=0}^{P} S_p T_p(s)\,. \end{aligned} \tag{23}$$

Employing the Chebyshev orthogonality relation

$$\int_{-1}^{1} T_m(s)\, T_p(s)\, \frac{ds}{\sqrt{1-s^2}} = \frac{\pi}{2}\, c_m\, \delta_{mp}\,, \tag{24}$$

where $c_0 = 2$, $c_m = 1$ for $m > 0$, insertion of (21) and (23) into (20) yields

$$\sum_{m=0}^{M} \omega_m b_m^{\pm} = 0\,, \tag{25}$$

p	$P = 64$	$P = 128$	$P = 256$	$P = 512$
0	0.432218	0.432304	0.432325	0.432331
1	0.270497	0.270627	0.270660	0.270668
2	-0.218246	-0.218075	-0.218032	-0.218021
3	-0.150727	-0.150597	-0.150564	-0.150556
4	-0.096500	-0.096328	-0.096285	-0.096275
5	-0.045275	-0.045144	-0.045111	-0.045103
6	-0.035763	-0.035591	-0.035548	-0.035537
7	-0.019975	-0.019843	-0.019811	-0.019803
8	-0.019032	-0.018860	-0.018817	-0.018806
9	-0.011571	-0.011439	-0.011406	-0.011398
10	-0.011993	-0.011819	-0.011776	-0.011766
11	-0.007633	-0.007501	-0.007468	-0.007460
12	-0.008308	-0.008133	-0.008090	-0.008079
13	-0.005450	-0.005316	-0.005284	-0.005276
14	-0.006128	-0.005951	-0.005908	-0.005898
15	-0.004109	-0.003974	-0.003941	-0.003933
16	-0.004729	-0.004551	-0.004508	-0.004497
17	-0.003224	-0.003088	-0.003055	-0.003047
18	-0.003778	-0.003598	-0.003555	-0.003544
19	-0.002610	-0.002473	-0.002440	-0.002431
20	-0.003102	-0.002920	-0.002876	-0.002866

Table 1: Chebyshev expansion coefficients E_p of the function $E(s)$ from (27) for $\lambda = 1$ and different values of the highest mode number P.

where

$$b_m^{\pm} = \begin{cases} \pm(\pi/2)\tanh(\lambda)\, C_m\, c_m & \text{for } m \text{ even} \\ (\pi/2)\, S_m & \text{for } m \text{ odd.} \end{cases} \tag{26}$$

Since $\exp(x) = \cosh(x) + \sinh(x)$, introducing the function

$$E(s) = (\pi/2)\,\Lambda\,\exp(\lambda s)\,\sqrt{1 - s^2} \approx \sum_{p=0}^{P} E_p T_p(s) \tag{27}$$

leads to

$$b_m^{\pm} = \begin{cases} \pm\tanh(\lambda)\, E_m\, c_m & \text{for } m \text{ even} \\ E_m & \text{for } m \text{ odd.} \end{cases} \tag{28}$$

The Chebyshev expansion coefficients E_m can be found by the standard descrete cosine transform of $E(s_i)$, where s_i are the Gauss-Lobatto points chosen according to the highest Chebyshev mode P. The results for the first 21 values of E_m from such transformations of $E(s)$ with $\lambda = 1$ and for various values of the P are shown in Table 1.

It is seen that the values of Chebyshev coefficients converge extremely poorly with increasing P. In figures 1a and 1b plots of $E(s)$ are shown for different values of λ.

Although $E(s)$ seems smooth throughout the interval $-1 \le s \le 1$, the function has infinite first derivatives at both end points. The sequence of E_m exibits an oscillation between even and odd terms, but multiplication of the even terms by $\tanh(\lambda)$ as required in (28) yields a monotonically decreasing sequence for sufficiently large values of m. This is seen in fig. 2, where we have plotted $m^2 E_m$ as well as $m^2 b_m^+$ versus m. As it is seen in this figure, and as it will be shown in the next section, the even and odd Chebyshev coefficients E_m each fall off asymptotically as m^{-2}, which limits the rate of convergence to $(1/P)^2$. This implies that determination of b_m by numerically performing the Chebyshev expansion in (27) is not useful.

1.3 Determination of $b_m^\pm$ by recursion relations

In section 1.2, we inserted the Chebyshev expansion of $\omega(s)$ from (18) into the integral solvability equation (17) and arrived at (20). Rearranging this equation, we find

$$\sum_{m=0}^{M} \omega_m \Big(\int_{-1}^{1} T_m(s) \sinh(\lambda s)\, ds \pm \tanh(\lambda) \int_{-1}^{1} T_m(s) \cosh(\lambda s)\, ds \Big) = 0\,. \tag{29}$$

¿From this expression we get

$$b_m^\pm = \begin{cases} \pm \tanh(\lambda)(\Lambda \int_{-1}^{1} T_m(s) \cosh(\lambda s)\, ds) & \text{for } m \text{ even} \\ \Lambda \int_{-1}^{1} T_m(s) \sinh(\lambda s)\, ds & \text{for } m \text{ odd.} \end{cases} \tag{30}$$

Here, we again have introduced the normalization constant $\Lambda = \lambda e^{-\lambda}$. It is clear that eq. (30) simply contains the analytical expressions for the Chebyshev coefficients that were determined numerically in sec. 1.2.

Introducing

$$\mathcal{E}_m = \Lambda \int_{-1}^{1} T_m(s) \exp(\lambda s)\, ds = \begin{cases} \Lambda \int_{-1}^{1} T_m(s) \cosh(\lambda s)\, ds & \text{for } m \text{ even} \\ \Lambda \int_{-1}^{1} T_m(s) \sinh(\lambda s)\, ds & \text{for } m \text{ odd,} \end{cases} \tag{31}$$

we can determine the asymptotic behavior of $b_m^\pm$ by analyzing $\mathcal{E}_m$. A variable substitution

$$s = \cos\theta \tag{32}$$

leads to

$$\mathcal{E}_m = \Lambda \int_{0}^{\pi} \cos(m\theta)\, h(\theta)\, ds\,, \tag{33}$$

where

$$h(\theta) = \exp(\lambda \cos\theta)\, \sin(\theta)\,. \tag{34}$$

Plots of $h(\theta)$ for various λ are shown in in fig. 3. The plots are shown for θ-values going slightly outside the interval $[0, \pi]$ in order to show that the function is smooth at the end points.

By repeated partial integration we obtain

$$\mathcal{E}_m = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{m^{2p}} \left[(-1)^m h^{2p-1}(\pi) - h^{2p-1}(0) \right]\,, \tag{35}$$

where $h^p(\theta_0)$ designates the p-th derivative of $h(\theta)$ at $\theta = \theta_0$. Since all the odd derivatives of $h(\theta)$ are non-zero at the endpoints of $[0, \pi]$, eq. (35) shows that $\mathcal{E}_m$ to leading order decays as m^{-2}.

Employing the basic Chebyshev polynomial relation

$$\int T_m(s)\, ds = \frac{T_{m+1}(s)}{2(m+1)} - \frac{T_{m-1}(s)}{2(m-1)} + \text{const} \quad \text{for } m = 2, 3, \ldots, \tag{36}$$

we find

$$\begin{aligned}
\Lambda \int_{-1}^{1} T_m(s) \exp(\lambda s)\, ds &= \Lambda \left[\left(\frac{T_{m+1}(s)}{2(m+1)} - \frac{T_{m-1}(s)}{2(m-1)} \right) \exp(\lambda s) \right]_{-1}^{1} \\
&\quad - \frac{\Lambda\lambda}{2(m+1)} \int_{-1}^{1} T_{m+1}(s) \exp(\lambda s)\, ds \\
&\quad + \frac{\Lambda\lambda}{2(m-1)} \int_{-1}^{1} T_{m-1}(s) \exp(\lambda s)\, ds \\
&\quad \text{for } m = 2, 3, \ldots\ ,
\end{aligned} \tag{37}$$

or

$$-\frac{\lambda}{2(m-1)}\, \mathcal{E}_{m-1} + \mathcal{E}_m + \frac{\lambda}{2(m+1)}\, \mathcal{E}_{m+1} = g_m \quad \text{for } m = 2, 3, \ldots, \tag{38}$$

where

$$g_m = \begin{cases} -\lambda(1 + e^{-2\lambda})/(m^2 - 1) & \text{for } m \text{ even} \\ -\lambda(1 - e^{-2\lambda})/(m^2 - 1) & \text{for } m \text{ odd.} \end{cases} \tag{39}$$

Thus, if we analytically calculate $\mathcal{E}_0$, $\mathcal{E}_1$, and $\mathcal{E}_2$ eq. (38) can in principal be used recursively to find $\mathcal{E}_m$ for $m = 3, 4, \ldots, M$. However, the absolute values of $\mathcal{E}_m$ determined by this procedure begin to grow very quickly after just a few recursions. (For $\lambda = 1$ this happens at $m = 14$.) The reason for this behavior is that the homogeneous recursion relation, obtained from (38) with $g_m = 0$, contains both a decaying and a growing solution. Any numerical error in determining $\mathcal{E}_m$ will trigger the growing solution, which eventually will take over. This problem is further accentuated by the oscillatory behavior of the forcing term g_m between even and odd values of m.

In order to formulate an accurate algorithm, we will adopt some of the procedures for homogeneous three-term recursion relations described by Gautschi in [1]. First, we will get rid of the oscillatory forcing, by only solving for $\mathcal{E}_m$ with even values of m. This can be accomplished by forming a linear combination of the three succesive equations corresponding to $2m-1$, $2m$ and $2m+1$. Introducing

$$\mathcal{C}_n = \mathcal{E}_{2n+2} = \Lambda \int_{-1}^{1} T_{2n+2}(s) \cosh(\lambda s)\, ds\,, \tag{40}$$

we then arrive at the three-term difference equation

$$\mathcal{C}_{n+1} + a_n\, \mathcal{C}_n + b_n\, \mathcal{C}_{n-1} = f_n \quad \text{for } n = 1, 2, \ldots, \tag{41}$$

where

$$
\begin{aligned}
a_n &= -\frac{4\,(2n+3)\,(2n+4)}{\lambda^2} - \frac{2\,(2n+4)}{2n+1} \\
b_n &= \frac{(2n+3)\,(2n+4)}{(2n+1)\,2n} \\
f_n &= \frac{4\,(2n+4)\,(1+e^{-2\lambda})}{\lambda\,(2n+1)} + \frac{6\,(1-e^{-2\lambda})}{(2n+1)\,n} \,.
\end{aligned}
\tag{42}
$$

Following [1] the asymptotic behavior of the homogeneous difference equation is determined by the asymptotic behavior of a_n and b_n as $n \to \infty$. From (42) we se that

$$
\begin{aligned}
a_n &\sim a n^{\alpha} = -\frac{16}{\lambda^2} n^2 \,, \\
b_n &\sim b n^{\beta} = 1 \,, \quad \text{for } n \to \infty.
\end{aligned}
\tag{43}
$$

This means that $\alpha = 2$ and $\beta = 0$, so after constructing the Newton-Puiseux diagram we find that the homogeneous difference equation obtained by setting $f_n = 0$ in (41) has two linearly independent solutions, $\mathcal{C}_{n,1}$ and $\mathcal{C}_{n,2}$, for which

$$
\begin{aligned}
\frac{\mathcal{C}_{n+1,1}}{\mathcal{C}_{n,1}} &\sim \frac{16}{\lambda^2} n^2 \,, \\
\frac{\mathcal{C}_{n+1,2}}{\mathcal{C}_{n,2}} &\sim \frac{\lambda^2}{16} n^{-2} \,, \quad \text{for } n \to \infty.
\end{aligned}
\tag{44}
$$

The solution $\mathcal{C}_{n,1}$ is called the 'dominant' solution while $\mathcal{C}_{n,2}$ is called the 'minimal' solution. Note that (44) describes the asymptotic behavior of the ratios between successive terms. If $\mathcal{C}_{n+1}/\mathcal{C}_n \sim n^{\gamma}$ then $\mathcal{C}_n \sim (n!)^{\gamma}$, which is a very rapid growth or decay, depending on whether γ is positive or negative. For $\gamma = 0$ we have $\mathcal{C}_{n+1}/\mathcal{C}_n \sim k$, so $\mathcal{C}_n \sim k^n$, indicating exponential growth or decay.

We are seeking a solution to the inhomogeneous difference equation (41) and we will do this in a standard way by first determining both of the solutions $\mathcal{C}_{n,1}$ and $\mathcal{C}_{n,2}$ to the homogeneous equation. Based on these solutions a particular solution will be formed by the method of variation of constants and finally a unique solution will formed by adjusting the particular solution by a linear combination of the homogeneous solutions in order to sastisfy the first values of the sequence $\mathcal{C}_n$ which will be determined analytically.

In the following a simpler notation will be used, where the minimal and dominant solutions will be denoted y_n and Y_n, respectively. In order to diminish the possibilities of numerical underflow or overflow for large values of n we introduce the ratios

$$
\begin{aligned}
r_n &= \frac{y_{n+1}}{y_n} \\
R_n &= \frac{Y_{n+1}}{Y_n} \,.
\end{aligned}
\tag{45}
$$

Following [1] we can rewrite the homogeneous difference equation for the minimal solution as

$$r_{n-1} = \frac{-b_n}{a_n + r_n} . \tag{46}$$

This equation can be solved by backwards recursion. We see that by doing this any error corresponding to triggering the dominant solution will quickly die out. Eventually, the boundary constraint coefficients, $b^{\pm}_m$, will be required for a finite number of Chebyshev modes M which corresponds to a maximum value N of the recursion index n. From (40) we see that

$$N = \frac{M-2}{2} , \tag{47}$$

The backwards recursion will have to be started at $n = \nu > N$. The precise value of ν will be left open for the moment. Given the asymptotic behavior of the minimal solution from (44) we set

$$r_\nu = \frac{\lambda^2}{16\nu^2} , \tag{48}$$

and use (46) to determine r_{n-1} for $n = \nu, \nu-1, \ldots, 1$. Since we are not interested in a particular minimal solution to the homogeneous difference equation we will leave the solution undetermined by the constant $y_{n=0}$ and let the ratios r_n, $n = 0, 1, \ldots, N-1$ determine the general form of the solution.

As for the dominant solution we write the homogeneneous difference equation in the form

$$R_n = -a_n - \frac{b_n}{R_{n-1}} \quad \text{for } n = 1, 2, \ldots. \tag{49}$$

Since the dominant and the minimal solution should be linearly independent, the starting value R_0 for this forward recursion relation is determined by requiring that the vector (Y_0, Y_1) is perpendicular to (y_0, y_1) which leads to $R_0 = -1/r_0$. As it will be explained in the following, eq. (49) should be used to determine R_n for all n up to ν.

We now construct a particular solution, z_n, to (41) as

$$z_n = A_n y_n + B_n Y_n , \tag{50}$$

where A_n and B_n will be determined by the method of variation of constants. First, we introduce

$$\begin{aligned} \Delta A_n &= A_{n+1} - A_n \\ \Delta B_n &= B_{n+1} - B_n . \end{aligned} \tag{51}$$

We then use the redundance in the determination of A_n and B_n to assume that

$$\Delta A_{n-1} y_n + \Delta B_{n-1} Y_n = 0 . \tag{52}$$

Substitution of (50) into (41) then leads to

$$\Delta A_{n-1} y_n r_n + \Delta B_{n-1} Y_n R_n = f_n . \tag{53}$$

Equations (52) and (53) can now be solved for $\Delta A_{n-1} y_n$ and $\Delta B_{n-1} Y_n$. Introducing the two parts of the particlar solution

$$\begin{aligned} u_n &= A_n y_n \\ v_n &= B_n Y_n \,, \end{aligned} \tag{54}$$

these solutions can be written as

$$\begin{aligned} u_n &= u_{n-1} r_{n-1} - \tau_n \\ v_n &= v_{n-1} R_{n-1} + \tau_n \,, \end{aligned} \tag{55}$$

where

$$\tau_n = \frac{f_n}{R_n - r_n} \,. \tag{56}$$

In principle the two recursion equations in (55) could be solved with arbitrary intial values u_0 and v_0 giving a particular solution z_n to (41) and a complete solution could then be formed by adding a linear combination of the homogeneous solutions $\alpha y_n + \beta Y_n$, where α and β are determined by the first two values of $\mathcal{C}_n$. However, we can just as well choose u_0 and v_0 in accordance with these two initial values so that z_n becomes not only a particular solution, but the complete solution to (41). Thus, we require that

$$\begin{aligned} \mathcal{C}_0 = z_0 &= u_0 + v_0 \\ \mathcal{C}_1 = z_1 &= u_1 + v_1 = u_0 r_0 + v_0 R_0 \,, \end{aligned} \tag{57}$$

which implies that

$$\begin{aligned} u_0 &= \frac{z_1 - R_0 z_0}{r_0 - R_0} \\ v_0 &= \frac{z_1 - r_0 z_0}{R_0 - r_0} \,, \end{aligned} \tag{58}$$

where

$$\begin{aligned} z_0 &= \Lambda \int_{-1}^{1} T_2(s) \cosh(s)\, ds \\ &= \left(1 - \frac{4}{\lambda} + \frac{4}{\lambda^2}\right) - \left(1 + \frac{4}{\lambda} + \frac{4}{\lambda^2}\right) e^{-2\lambda} \,, \end{aligned} \tag{59}$$

and

$$\begin{aligned} z_1 &= \Lambda \int_{-1}^{1} T_4(s) \cosh(s)\, ds \\ &= \left(1 - \frac{16}{\lambda} + \frac{80}{\lambda^2} - \frac{192}{\lambda^3} + \frac{192}{\lambda^4}\right) \\ &\quad - \left(1 + \frac{16}{\lambda} + \frac{80}{\lambda^2} + \frac{192}{\lambda^3} + \frac{192}{\lambda^4}\right) e^{-2\lambda} \,. \end{aligned} \tag{60}$$

The determination of u_n from (55) with u_0 given by (58-60) poses no problem. However, the recursion relation for v_n in (55) contains R_n, which from (44) is seen to grow asymptotically like n^2. Since the full solution z_n is asymptotically decaying and u_n is decaying, then v_n must also asymptotically decay. However, this can only be the case for a very delicate balance between terms in the equation for v_n in (55), and any error will immediately blow up. In order to find v_n in a numerically stable matter, we must solve by backwards recurence as in the case of the minimal solution to the homogeneous problem. Thus, we write

$$v_{n-1} = \frac{v_n - \tau_n}{R_{n-1}}\,, \tag{61}$$

and start the solution by setting $v_\nu = 0$. Here $\nu > N$ has the same value as in the solution for y_n. After determining this particular solution to the two-term recursion relation for v_n in (55) we must add a multiple aY_n of the homogenenous solution to obtain a complete solution. However, since Y_n is a (rapidly) growing function, we must set $a = 0$. This, in turn, means that v_0 determined through backwards recursion of (61) automatically should obtain the correct value given by (58-60). In figure 4 we show the dependence of v_n determined through (61) on the start value v_ν. Each sequence of v_n-values are connected by straight lines to guide the eye. The large circles indicates the soltion obtained by setting $v_{100} = 0$ while the sequences of small circles show the solutions for other initial values. It is clearly seen that even if v_ν is given a value very far from value obtained by setting $v_{100} = 0$, all sequences converge the same v_n for small n. A discussion of how an appropriate value for ν for which $v_\nu = 0$ is chosen is contained at the end of this section.

We have now found the values of $\mathcal{C}_n = \mathcal{E}_{2n+2}$ for $n = 0, 1, \ldots, (M-2)/2$. The value of $\mathcal{E}_0$ is easily found to be

$$\mathcal{E}_0 = \Lambda \int_{-1}^{1} T_0(s)\cosh(\lambda s)\, ds = 1 - e^{-2\lambda}\,, \tag{62}$$

so all the values of $b_m^\pm$ in (30) for even values of m can now be found by multyplying the even indexed values of $\mathcal{E}_m$ by $\pm\tanh(\lambda)$. As for the odd indexed values of $b_m^\pm$ we first introduce

$$\mathcal{S}_n = \mathcal{E}_{2n+1} = \Lambda \int_{-1}^{1} T_{2n+1}(s)\sinh(\lambda s)\, ds\,. \tag{63}$$

Employing the basic Chebyshev polynomial relation (36) we find by partial integration that

$$\begin{aligned} \mathcal{S}_n &= \left(\frac{\lambda}{4n+4} - \frac{\lambda}{4n}\right)\left(1 - e^{-2\lambda}\right) - \frac{\lambda}{4n+4}\mathcal{C}_n + \frac{\lambda}{4n}\mathcal{C}_{n-1} \\ &\quad \text{for } n = 1, 2, \ldots, (M-2)/2. \end{aligned} \tag{64}$$

expressing that $\mathcal{S}_n$ can be determined directly from the previously found values of $\mathcal{C}_n$. Direct integration gives

$$\mathcal{S}_0 = \left(1 - \frac{1}{\lambda}\right) + \left(1 + \frac{1}{\lambda}\right)e^{-2\lambda}\,, \tag{65}$$

λ	N	ν	Δv_0	$\mathcal{C}_N$	$\mathcal{S}_N$
0.1	10	13	10^{-13}	$-6 \cdot 10^{-4}$	$-5 \cdot 10^{-5}$
0.1	50	53	10^{-13}	$-2 \cdot 10^{-5}$	$-2 \cdot 10^{-6}$
0.1	100	102	10^{-13}	$-5 \cdot 10^{-6}$	$-5 \cdot 10^{-7}$
1	100	103	10^{-16}	$-3 \cdot 10^{-5}$	$-2 \cdot 10^{-5}$
1	500	502	10^{-16}	$-1 \cdot 10^{-6}$	$-8 \cdot 10^{-7}$
1	1000	1002	10^{-16}	$-3 \cdot 10^{-7}$	$-2 \cdot 10^{-7}$
10	100	105	$5 \cdot 10^{-17}$	$-3 \cdot 10^{-4}$	$-3 \cdot 10^{-4}$
10	500	504	$5 \cdot 10^{-17}$	$-1 \cdot 10^{-5}$	$-1 \cdot 10^{-5}$
10	1000	1003	$5 \cdot 10^{-17}$	$-3 \cdot 10^{-6}$	$-3 \cdot 10^{-6}$
100	100	112	10^{-16}	$-3 \cdot 10^{-3}$	$-3 \cdot 10^{-3}$
100	500	506	10^{-16}	$-1 \cdot 10^{-4}$	$-1 \cdot 10^{-4}$
100	1000	1005	10^{-16}	$-3 \cdot 10^{-5}$	$-3 \cdot 10^{-5}$
1000	500	520	$5 \cdot 10^{-17}$	$-1 \cdot 10^{-3}$	$-1 \cdot 10^{-3}$
1000	1000	1013	$5 \cdot 10^{-17}$	$-3 \cdot 10^{-4}$	$-3 \cdot 10^{-4}$
1000	2000	2009	$5 \cdot 10^{-17}$	$-6 \cdot 10^{-5}$	$3 \cdot 10^{-5}$

Table 2: Values of initial index ν for backward recurrences in order to obtain IEEE double precision machine number accuracy of the iterations. λ is the normalized Fourier mode index defined in (4), $N = (M-2)/2$ is the highest index in the sequences for $\mathcal{C}_n$ and $\mathcal{S}_n$, M is the highest index of $b_m^{\pm}$, Δv_0 is the absolute difference between the values of v_0 determined by by (61) and (58-60), respectively.

which completes the calculation of $b_m^{\pm}$.

The value of ν is determined as follows. We start by setting $\nu = N$ and find the solutions $\mathcal{C}_n^1$ and $\mathcal{S}_n^1$. We then increase the value of ν by 1 and find $\mathcal{C}_n^2$ and $\mathcal{S}_n^2$. An 'error', δ, is now defined as $\delta = \max(|\mathcal{C}_n^2 - \mathcal{C}_n^1|, |\mathcal{S}_n^2 - \mathcal{S}_n^1|)$ in the interval $n = 1, 2, \ldots, N$. If the δ is less than a prescribed level, ϵ, $\mathcal{C}_n^2$ and $\mathcal{S}_n^2$ are chosen as solutions. Otherwise, the value of ν is increased by 1 and the procedure is repeated. We have determined values of ν by this iterative method for various values of λ, $N = (M-2)/2$ and ϵ. It turns out, that the iterations converge very rapidly, and that it is easy to obtain values for δ that are smaller than the smallest double precision machine number resulting in $\delta = 0$. At the same time, the absolute difference Δv_0 between the values of v_0 determined by (61) and (58-60) is very small and seems within the range of accuracy by which the analytical value from (58-60) is determined. In Table 2 we summerize some of the results of these investigatons. Here we list the values of ν and Δv_0 for which double precision machine accuracy in the iterations were obtained for various values of λ and N. Also listed are the values obtained for $\mathcal{C}_N$ and $\mathcal{S}_N$. Note that contrary to the case described in the previous section for determination of $b_m^{\pm}$ by numerical Chebyshev expansions, high accuracy in the determination of $b_m^{\pm}$ is obtained irrespective of the truncation value M (or P in (27)). ¿From Table 2 it is seen to be fully sufficient to set $\nu = N + 25$ for accurate determination of $b_m^{\pm}$ for values of λ and N of relevance to our problem.

1.4 Determination of $b_m^\pm$ by modified Green's functions

In our previous work, the boundary constraint coefficients, $b_m^\pm$, were found by modified Green's functions. These are response functions for the Diriclet Poisson equation to Chebyshev polynomial forcing, rather than the delta function forcing of standard Green's functions. Thus, the modified Green's functions, $\sigma_m(y)$, satisfy

$$\frac{d^2\sigma_m}{dy^2} - \lambda^2\sigma_m = -T_m(y)\,, \tag{66}$$

and

$$\sigma_m(\pm 1) = 0\,. \tag{67}$$

In our old paper the constraints are then found as

$$b_m^\pm = \left.\frac{d\sigma_m}{dy}\right|_{\pm 1}\,, \tag{68}$$

and a numerical scheme was described to calculate the values of $b_m^\pm$ from this expression.

Here, we will show an analytical expression for $\sigma_m(y)$ and give an exact value for $b_0^\pm$. First, we note that a solution to the homogeneous Poisson equation is

$$\sigma_m^h = Ae^{\lambda y} + Be^{-\lambda y}\,. \tag{69}$$

A particular solution can be then found by variation of constants, so

$$\sigma_m^p = A(y)e^{\lambda y} + B(y)e^{-\lambda y}\,. \tag{70}$$

By substituting σ_m^p from (70) into (66) and by making the usual assumption

$$A_y e^{\lambda y} + B_y e^{-\lambda y} = 0\,, \tag{71}$$

we find

$$A_y e^{\lambda y} - B_y e^{-\lambda y} = -\frac{T_m(y)}{\lambda}\,. \tag{72}$$

By solving eqs. (71) and (72) for A_y and B_y and integrating with respect to y we find

$$\begin{aligned} A(y) &= -\frac{1}{2\lambda}\int T_m(y)\, e^{-\lambda y}\, dy \\ B(y) &= \frac{1}{2\lambda}\int T_m(y)\, e^{\lambda y}\, dy\,, \end{aligned} \tag{73}$$

so a complete solution to the modified Green's function is

$$\sigma_m(y) = \left(-\frac{1}{2\lambda}\int T_m(y)\, e^{-\lambda y}\, dy\right)\, e^{\lambda y} + \left(\frac{1}{2\lambda}\int T_m(y)\, e^{\lambda y}\, dy\right)\, e^{-\lambda y} + A_0\, e^{\lambda y} + B_0\, e^{-\lambda y}\,, \tag{74}$$

where the constants A_0 and B_0 are determined by the boundary conditions in (67).

For $m = 0$ we get

$$\sigma_0(y) = \frac{1}{\lambda^2}\left[1 - \frac{\cosh(\lambda y)}{\cosh(\lambda)}\right], \tag{75}$$

which leads to

$$b_0^{\pm} = \left.\frac{d\sigma_0}{dy}\right|_{\pm 1} = \mp\frac{1}{\lambda}\,\frac{1 - e^{-2\lambda y}}{1 + e^{-2\lambda y}}\,. \tag{76}$$

This value is found to be in excellent agreement with our previous, numerically determined values.

References

[1] W. Gautschi (1967). *Computational Aspects of Three-Term Recurrence Relations*, SIAM Review **9**, 24–82.