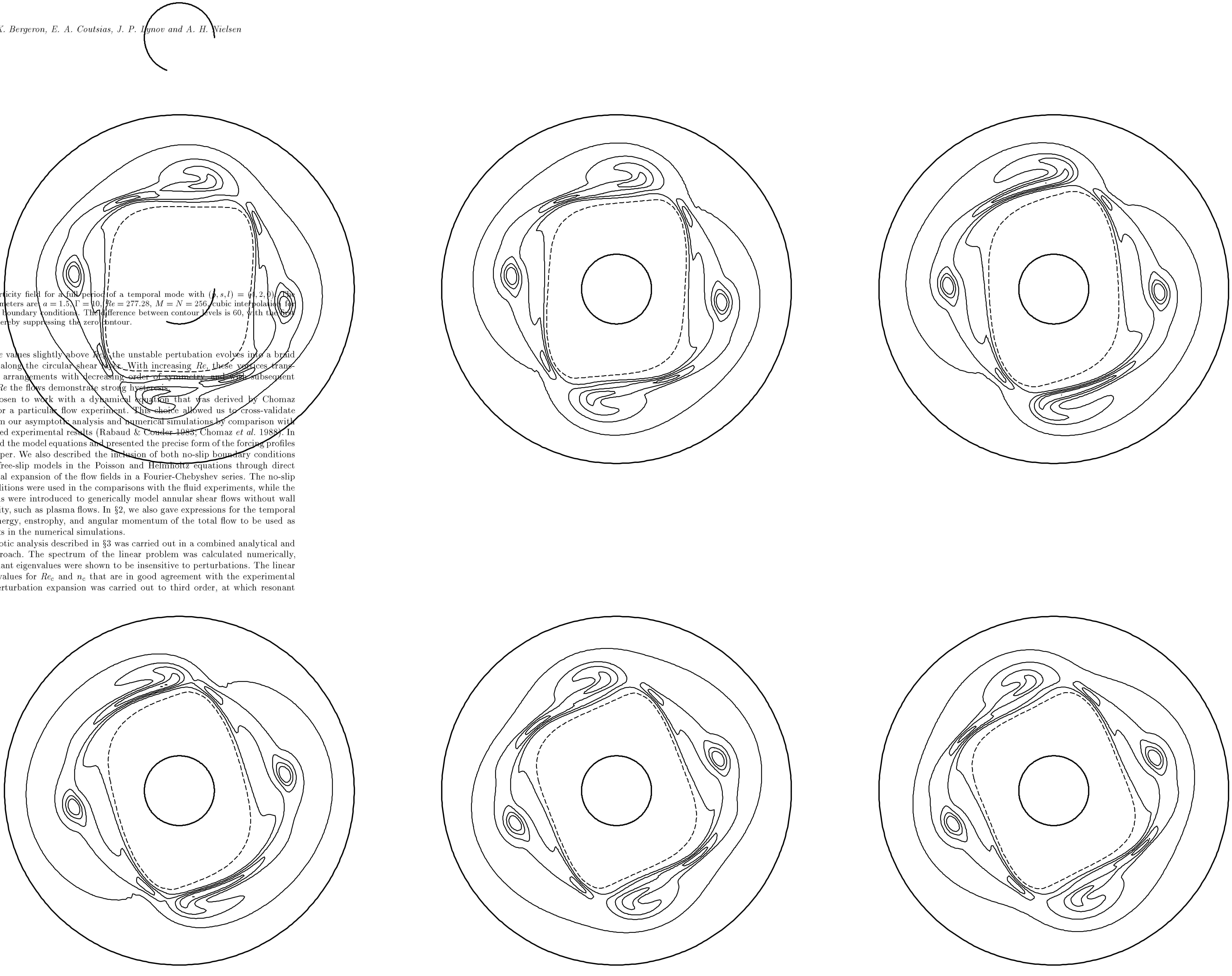


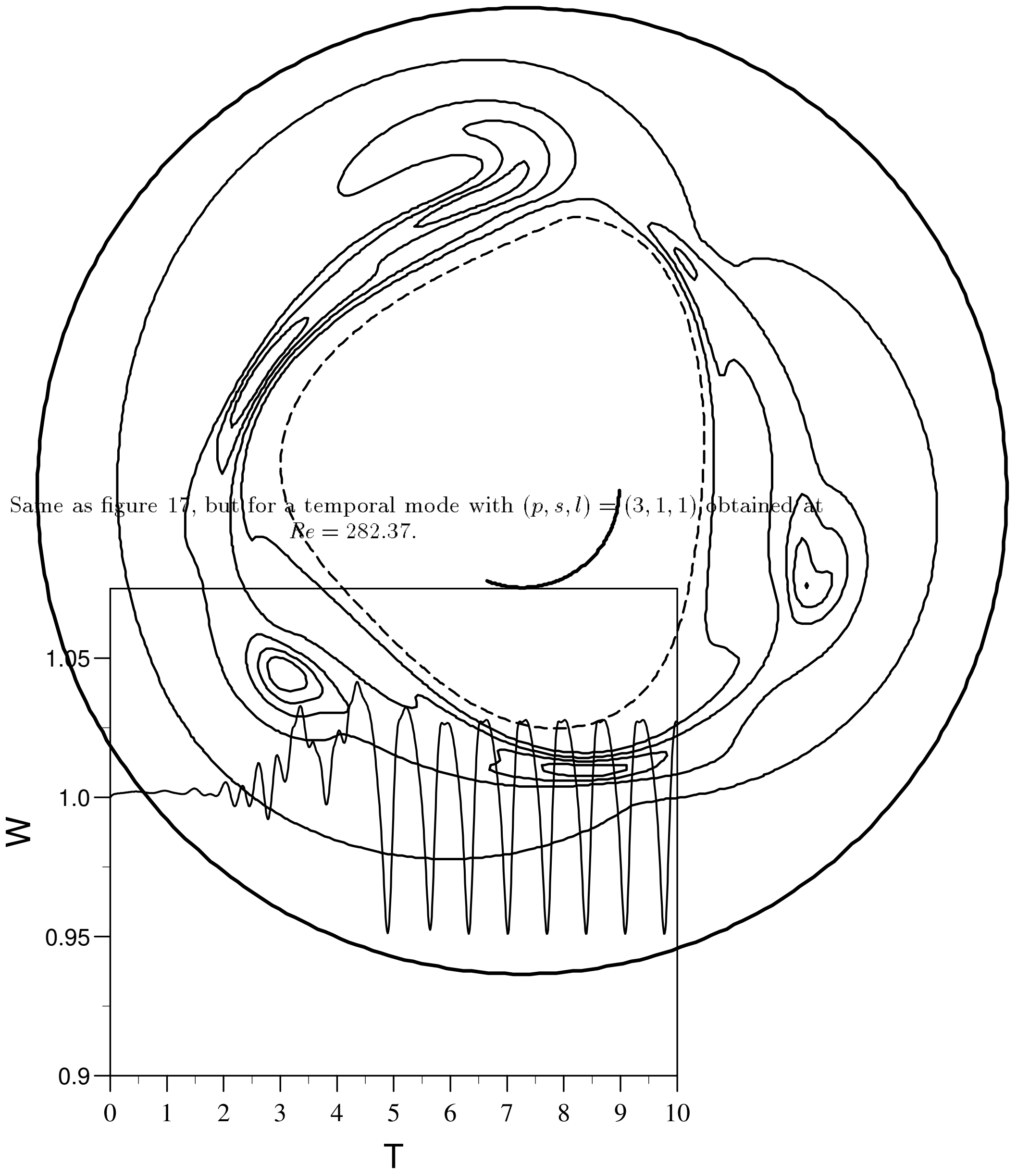
FIGURE 17. Vorticity field for a full period of a temporal mode with $(p, s, l) = (4, 2, 0)$. The simulation parameters are: $a = 1.5$, $\Gamma = 10$, $Re = 277.28$, $M = N = 256$, cubic interpolation for Ω^* and free-slip boundary conditions. The difference between contour levels is 60, with the first levels at ±30 thereby suppressing the zero contour.

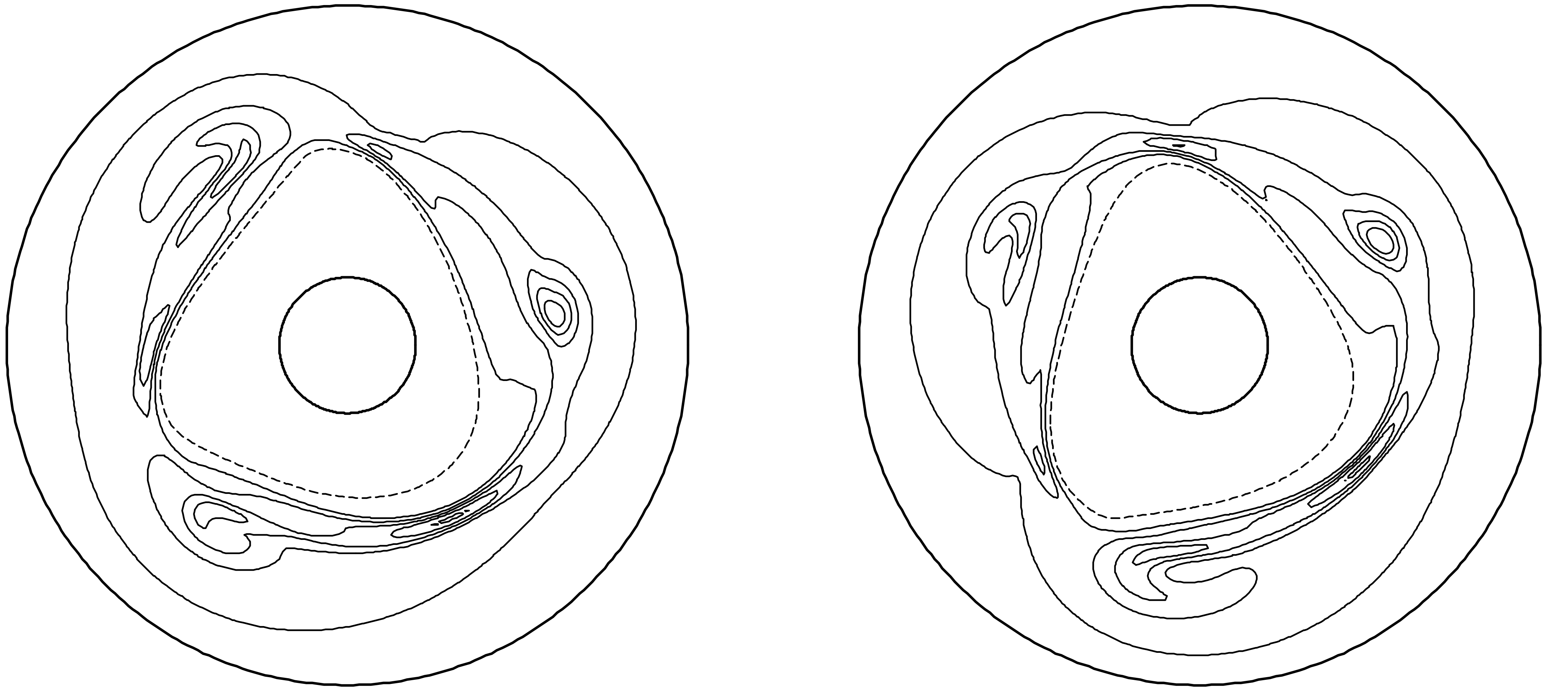
ber, n_c. For Re values slightly above Re_c the unstable pertubation evolves into a braid of n_c vortices along the circular shear layer. With increasing Re, these vortices transform into new arrangements with decreasing order of symmetry, and with subsequent decreasing of Re the flows demonstrate strong hysteresis.

We have chosen to work with a dynamical equation that was derived by Chomaz *et al.* (1988) for a particular flow experiment. This choice allowed us to cross-validate the results from our asymptotic analysis and numerical simulations by comparison with well-documented experimental results (Rabaud & Couder 1983; Chomaz *et al.* 1988). In §2, we described the model equations and presented the precise form of the forcing profiles used in our paper. We also described the inclusion of both no-slip boundary conditions and different free-slip models in the Poisson and Helmholtz equations through direct use of a spectral expansion of the flow fields in a Fourier-Chebyshev series. The no-slip boundary conditions were used in the comparisons with the fluid experiments, while the free-slip models were introduced to generically model annular shear flows without wall induced vorticity, such as plasma flows. In §2, we also gave expressions for the temporal evolution of energy, enstrophy, and angular momentum of the total flow to be used as accuracy checks in the numerical simulations.

The asymptotic analysis described in §3 was carried out in a combined analytical and numerical approach. The spectrum of the linear problem was calculated numerically, and the dominant eigenvalues were shown to be insensitive to perturbations. The linear analysis gave values for Re_c and n_c that are in good agreement with the experimental values. The perturbation expansion was carried out to third order, at which resonant

FIGURE 18. Same as figure 17, but for a temporal mode with $(p, s, l) = (3, 1, 1)$, obtained at $Re = 282.37$.





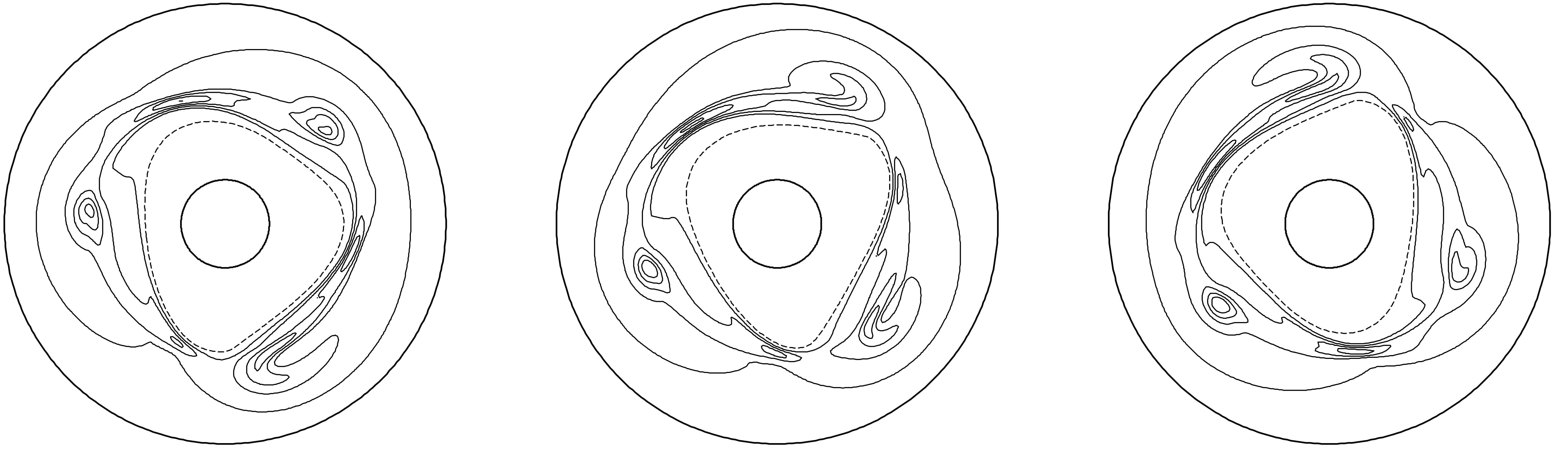


FIGURE 19. The evolution of global enstrophy, W, in a temporal state with both azimuthal mode $n = 4$ and $n = 2$ present. The evolution of vorticity field in the oscillatory phase is shown in figure 17.

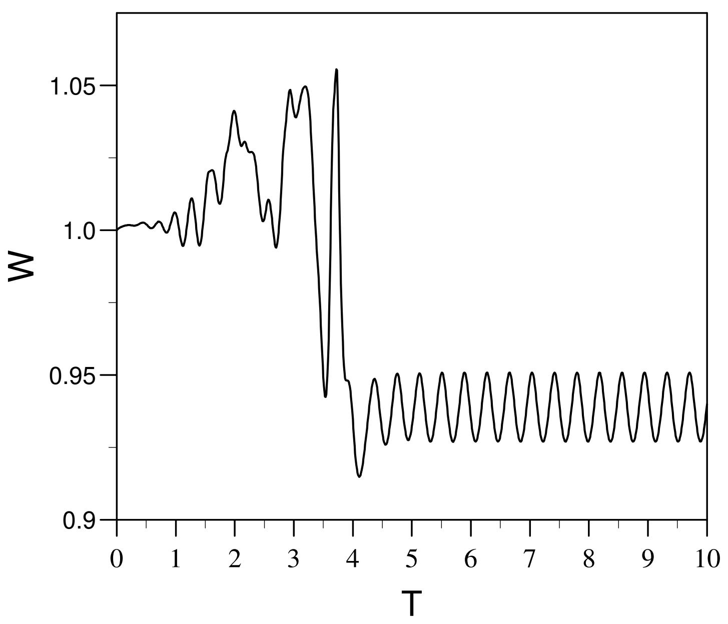


FIGURE 20. The evolution of global enstrophy, W, in a temporal state with both azimuthal mode $n = 3$ and $n = 1$ present. The evolution of vorticity field in the oscillatory phase is shown in figure 18.

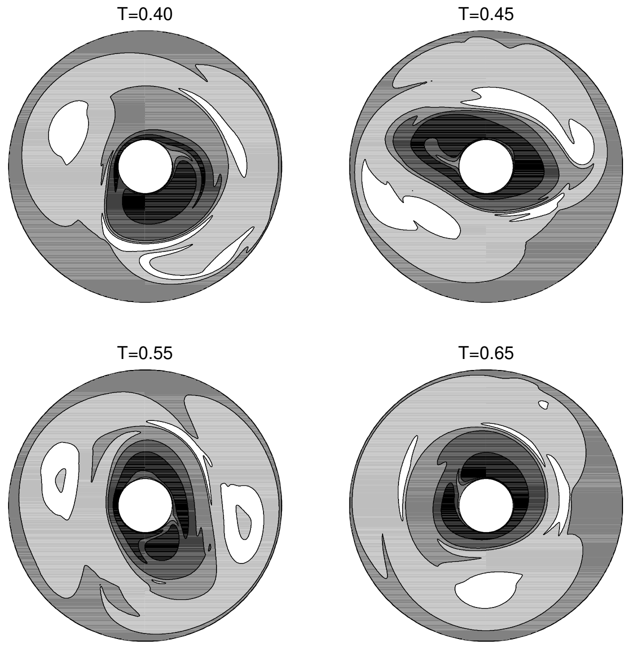


FIGURE 21. The evolution of the vorticity field at $Re = 1528$ with free-slip boundary conditions. Dark regions represent low levels of vorticity and white regions high levels. A cubic profile was used in the angular velocity forcing, $a = 1.5$, $\Gamma = 10$, and $\varepsilon = 0$.

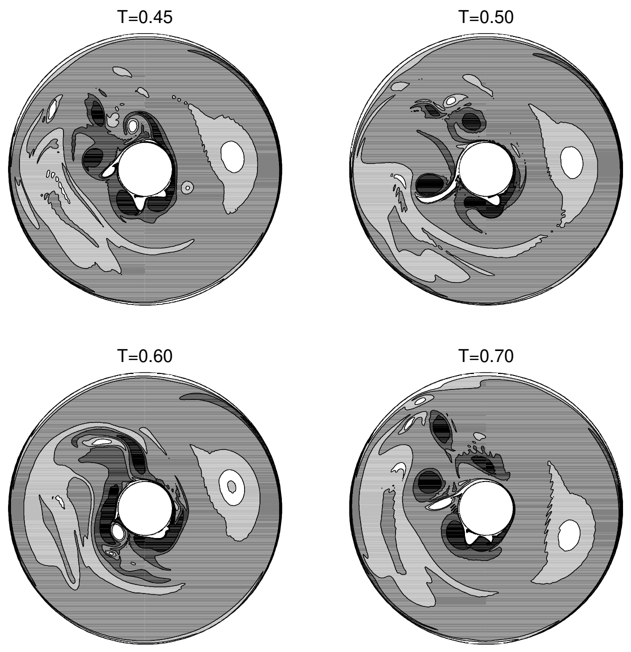


FIGURE 22. Same as figure 21, but with no-slip boundary conditions.

terms appear. Application of the Fredholm alternative theorem lead to a Landau equation describing the saturation behaviour of the neutral mode. Unfortunately, the variational problems occurring at the second and third order levels are numerically ill-conditioned so direct numerical solutions do not give accurate results. As mentioned in §3, we are presently testing different conditioning methods to make these problems solvable with high accuracy.

The spectral method used in our numerical simulations was briefly described in §4 and results from the simulations presented in §5. Our first simulations were focused on the near linear problem. Here, we demonstrated a clear supercritical bifurcation behaviour for flows subjected to both no-slip and free-slip boundary conditions, and very close agreement was obtained when comparing the results from the full simulations with those from the linear analysis. When comparing with the results from fluid experiments, we obtained a much better agreement than previously obtained by others, and we ascribe this close agreement to the high accuracy of our numerical solution. While the quantitative agreement between the experimental results and the results from previous numerical studies by others were so rough that the validity of the model equations could not be definitely confirmed, the close agreement we have presented in this paper demonstrates that the model equations are fully adequate for describing the experimentally observed flow dynamics, at least for low to moderate *Re*-values. In particular, our results show that three-dimensional effects are not important for these fluid flows.

The temporal evolution of both the total energy and the total enstrophy in the system during a transition away from the axisymmetric state was monitored, and clear drops in both quantities were observed. This indicates why the new state is preferred. We are able to monitor the temporal evolution of the content in each azimuthal mode of both energy and enstrophy, and time traces of these modal quantities have given us accurate information on how the various modes either grow and saturate, or die away. By following the growth and saturation of a perturbation corresponding to the most unstable mode, we were able to determine the complex Landau coefficients, and using these coefficients we demonstrated that the Landau equation models the temporal behaviour of the mode saturation obtained by the full simulation very closely.

Numerical studies of the behaviour of the flow during spin-up and spin-down were presented in §5.2. In an attempt to give a quantitative description of the flow evolution both during transitions and in between, we again found it very useful to follow the evolution of the total enstrophy, or energy, as well as the distribution of the enstrophy, or energy, on the different azimuthal modes. Although the precise dynamical evolution of the flow, including the details of the symmetry breaking bifurcations, depends on a number of different parameters, we were able to extract some generic properties which characterise the system even far away from equilibrium during both spin-up and spin-down. From time traces of the total enstrophy, or energy, we observed that the adjustment of the flow to changes in *Re* occurs on two very different time scales. Which of the two time scales is relevant depends on whether the flow evolves with unchanged symmetry or not.

As long as the symmetry (number of vortices) is constant, the adjustment of the flow to changes in *Re* is very rapid (almost instantaneous), at least much faster than the time scales related to the spin-up and spin-down rates used in our simulations. This gives rise to unique relations, independent of history, between global quantities (such as E and W) in a flow with a given number of vortices and *Re*, even in cases when *Re* is changing rapidly in time. The values of the global quantities in these 'states of attraction' for given n and *Re* appear to be unique and insensitive to noise for a particular flow geometry, even for flows in which the azimuthal symmetry is kept artificially constant.

On the other hand, during transition from one symmetry type to another the flow evolves on a much longer time scale. During spin-up, the transition process leads to a reduction in the number of vortices, and we have shown that this is accompanied by clear drops in both E and W. The onset of the instability that causes the symmetry breaking transitions is not easily determined. Adding noise to the forcing function or increasing the spin-up rate both have the effect of lowering the value of *Re* at which transition occurs. If a transition during spin-up is postponed for some reason, for example by spinning up very rapidly, the number of vortices in the flow may be reduced by more than one when a transition finally occurs.

A drop in E and W seems to be the driving mechanism behind all the mode transitions during both spin-up and spin-down. Although it is not possible to separate the drops in E from those in W, as they occur simultaneously, we find it most natural if a minimization of energy is the main cause for the transitions. We have seen, however, that although a mode transition is possible from an energy minimization viewpoint, it may not occur. Especially during spin-down the flow seems very reluctant to make a mode transition and only by adding azimuthal perturbations to the noise function were we able to trigger a transition. We would like to emphasize, that when these distinctions between the effects of various types of external perturbations are possible, at all, it is due to the very high accuracy of our spectral method.

We also addressed the properties of two-dimensional shear flows at high Reynolds numbers, where temporal states are formed. Simulations revealed the existence of both regular temporal states, in which a few modes are simultaneously present, and states with chaotic temporal behaviour. Contrary to the flows at lower *Re*, these high Reynolds states are very sensitive, also qualitatively, to changes in boundary conditions. It can be expected that modelling real experiments at high values of *Re* will require extensions to the model equations used in this paper. The precise way in which the model equations should be modified will require a detailed physical description of the particular set-up, and we anticipate that the results from such simulations will be less general than the ones presented here.

During the course of work reported in this paper, two of the authors (KB and EAC) visited Risø National Laboratory and another of the authors (JPL) visited the University of New Mexico - the hospitality of both institutions during these visits is gratefully acknowledged. The authors are also grateful to J.S. Hesthaven for many valuable discussions on various aspects of spectral methods. This work was supported by DOE Grant DE-FG03-92ER25128 and by the Danish Research Councils.

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