Flat complex connections with torsion of type $(1,1)$

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Outline

Given an almost complex manifold (M, J), we will be interested in studying complex connections ∇ on M with trivial holonomy and such that:

- $\nabla J = 0$,
- the torsion T is of type $(1,1)$ with respect to J.

Let ∇ be an affine connection on a manifold M with torsion tensor field T, where $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$, for all X, Y vector fields on M.

Given an almost complex structure J on M the torsion T of a connection ∇ on the almost complex manifold (M,J) is said to be:

- of type $(1,1)$ if $T(JX,JY) = T(X,Y)$,
- of type $(2,0)$ if $T(JX,Y) = JT(X,Y)$,

for all vector fields X, Y on M.

∇ is a complex connection when $\nabla J = 0$.

Examples of complex connections: in a Hermitian manifold (M, J, g), consider ∇^1 and ∇^2 defined by

$$g\left(\nabla^1_X Y, Z\right) = g\left(\nabla^g_X Y, Z\right) + \frac{1}{4}\left(d\omega(X, JY, Z) + d\omega(X, Y, JZ)\right),$$

$$g\left(\nabla^2_X Y, Z\right) = g\left(\nabla^g_X Y, Z\right) - \frac{1}{2} d\omega(JX, Y, Z),$$

where $\omega(X, Y) := g(JX, Y)$ is the Kähler form corresponding to g and J. These connections satisfy

$$\nabla^1 g = 0, \quad \nabla^1 J = 0, \quad T^1 \text{ is of type } (1,1),$$

$$\nabla^2 g = 0, \quad \nabla^2 J = 0, \quad T^2 \text{ is of type } (2,0),$$

These are the *first and second canonical Hermitian connections*.

Remark

- ∇^1 *can also be written as* $\nabla^1_X Y = \frac{1}{2}\left(\nabla^g_X Y - J\nabla^g_X JY\right)$.
- ∇^2 *is also known as the Chern connection; it is unique satisfying the conditions above.*

We denote by N the Nijenhuis tensor of J, defined by

$$N(X, Y) := [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y].$$

Newlander-Nirenberg ('57): J is integrable if and only if $N \equiv 0$.

Lemma

Let (M, J) be an almost complex manifold.

(i) *If $\nabla J = 0$ and T is of type $(1,1)$ then J is integrable.*

(ii) *If $\nabla J = 0$ and T is of type $(2,0)$, then J is integrable.*

Complex connections with trivial holonomy

Let M be an n-dimensional connected manifold and ∇ an affine connection on M with trivial holonomy (hence flat). Then (Hicks '59 - Wolf '72):

- the space $\mathcal{P}^\nabla$ of parallel vector fields on M is an n-dimensional real vector space;
- $T(X,Y) = -[X,Y], \qquad$ for all $X, Y \in \mathcal{P}^\nabla$;
- $\mathcal{P}^\nabla$ is a Lie algebra if and only if T is parallel.

Lemma

Let (M,J), be a connected almost complex manifold with an affine connection ∇ on M, Hol (∇) trivial. Then the following conditions are equivalent:

(i) $\nabla J = 0$;

(ii) *the space $\mathcal{P}^\nabla$ of parallel vector fields is J-stable;*

(iii) *there exist parallel vector fields $X_1, \dots, X_n, JX_1, \dots, JX_n$, linearly independent at every point of M.*

Complex parallelizable manifolds

We recall that a complex manifold (M, J) is called *complex parallelizable* when there exist n holomorphic vector fields $Z_1, \dots, Z_n$, linearly independent at every point of M.

The following classical result, due to Wang ('54), characterizes the compact complex parallelizable manifolds.

Theorem

Every compact complex parallelizable manifold may be written as a quotient space $\Gamma \backslash G$ of a complex Lie group by a discrete subgroup Γ.

We show next a result which relates the notion of complex parallelizability with the existence of a flat complex connection with torsion of type $(2, 0)$.

Proposition

Let M be a connected $2n$-dimensional manifold with a complex structure J. Then the following conditions are equivalent:

(i) *there exist vector fields $X_1, \dots, X_n, JX_1, \dots, JX_n$, linearly independent at every point of M, such that*

$$[X_k, X_l] = -[JX_k, JX_l], \;\; k < l, \quad [JX_k, X_l] = J[X_k, X_l], \;\; k \leq l,$$

(ii) *there exist n holomorphic vector fields $Z_1, \dots, Z_n$ which are linearly independent at every point of M (in other words, (M, J) is complex parallelizable);*

(iii) *there exists a complex connection ∇ on M with trivial holonomy whose torsion tensor field T is of type $(2, 0)$.*

Corollary

Let (M, J) be a connected complex manifold and ∇ an affine connection with trivial holonomy. Then ∇ satisfies (iii) in the previous proposition if and only if the space $\mathcal{P}^\nabla$ of parallel vector fields is J-stable and J satisfies

$$J[X, Y] = [X, JY] \quad \text{for any} \quad X, Y \in \mathcal{P}^\nabla.$$

Flat complex connections with $(1,1)$-torsion

Flat complex connections with $(1,1)$-torsion

Proposition

Let M be a connected $2n$-dimensional manifold with an almost complex structure J. Then the following conditions are equivalent:

(i) *there exist vector fields $X_1, \ldots, X_n, JX_1, \ldots, JX_n$, linearly independent at every point of M, such that*

$$[X_k, X_l] = [JX_k, JX_l], \qquad [JX_k, X_l] = -[X_k, JX_l], \qquad k < l;$$

(ii) *there exist n commuting complex vector fields $Z_1, \ldots, Z_n$ which are linearly independent sections of $T^{1,0}M$ at every point of M;*

(iii) *there exists a complex connection ∇ on M with trivial holonomy whose torsion tensor field T is of type $(1,1)$.*

Moreover, any of the above conditions implies that J is integrable.

Definition

An affine connection ∇ *on a connected almost complex manifold* (M, J) *will be called an* abelian *connection if it satisfies condition (iii) of previous Proposition.*

Corollary

Let (M, J) *be a connected complex manifold and* ∇ *an affine connection with trivial holonomy. Then* ∇ *is an abelian connection on* (M, J) *if and only if the space* $\mathcal{P}^{\nabla}$ *of parallel vector fields is J-stable and J satisfies*

$$[JX, JY] = [X, Y] \quad \text{for any} \quad X, Y \in \mathcal{P}^{\nabla}.$$

Examples

We consider:

1. A connected Lie group G,
2. a complex structure J on its Lie algebra $\mathfrak{g}$ ($\longleftrightarrow$ a left invariant complex structure on G),
3. a $\mathfrak{g}$-valued bilinear form $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ ($\longleftrightarrow$ a left invariant connection on G),
4. a compatibility condition $\nabla_x Jy = J\nabla_x y$, $x, y \in \mathfrak{g}$ ($\longleftrightarrow$ $\nabla J = 0$ on G).

If $\Gamma \subset G$ is any discrete subgroup of G, then J induces a complex structure on the quotient $\Gamma\backslash G$, which will be denoted J_0.

The left invariant affine connection ∇ on G defined by $\nabla_X Y = 0$ for all X, Y left invariant vector fields on G is known as the $(-)$-connection. This connection satisfies:

1. $T(X, Y) = -[X, Y]$ for all X, Y left invariant vector fields on G;
2. $\nabla T = 0$ and $\mathcal{P}^{\nabla} = \mathfrak{g} \subset \mathfrak{X}(G)$;
3. $\mathrm{Hol}(\nabla)$ is trivial, thus, ∇ is flat;
4. The geodesics of ∇ through the identity $e \in G$ are Lie group homomorphisms $\mathbb{R} \to G$, therefore, ∇ is complete;
5. The parallel transport along any curve joining $g \in G$ with $h \in G$ is given by the derivative of the left translation $(dL_{hg^{-1}})_g$.

If $\Gamma \subset G$ is any discrete subgroup of G, then this connection induces a complete connection on the quotient $\Gamma\backslash G$, denoted ∇^0, with parallel torsion.

If J is a left invariant complex structure on G, then J is parallel with respect to the $(-)$-connection ∇. Hence

$(\Gamma \backslash G, J_0)$ carries a complete complex connection ∇^0 with trivial holonomy and parallel torsion.

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$(\Gamma\backslash G, J_0)$ carries a complete complex connection ∇^0 with trivial holonomy and parallel torsion.

In the next result we show that the converse also holds.

Theorem

The triple (M, J, ∇) where M is a connected manifold endowed with a complex structure J and a complex connection ∇ with trivial holonomy is equivalent to a triple $(\Gamma\backslash G, J_0, \nabla^0)$ as above if and only if ∇ is complete and its torsion is parallel.

Idea of the proof: a suitable modification of the proof of the Cartan-Ambrose-Hicks Theorem.

Note that the $(-)$-connection has torsion of type $(1,1)$ with respect to a complex structure J if and only if

$$[Jx, Jy] = [x, y] \qquad \forall x, y \in \mathfrak{g}.$$

A complex structure on a Lie algebra satisfying this condition is called *abelian*. They were introduced by Barberis-Dotti-Miatello in '95 and much studied since then.

Corollary

Let ∇ be an abelian connection on a connected complex manifold (M, J) such that the torsion tensor field T is parallel. Then:

(i) *the space $\mathcal{P}^{\nabla}$ of parallel vector fields on M is a Lie algebra and J is an abelian complex structure on $\mathcal{P}^{\nabla}$;*

(ii) *the Lie algebra $\mathcal{P}^{\nabla}$ is 2-step solvable.*

(iii) *if, furthermore, ∇ is complete, then (M, J, ∇) is equivalent to $(\Gamma\backslash G, J_0, \nabla^0)$, where G is a simply connected 2-step solvable Lie group equipped with a left invariant abelian complex structure and $\Gamma \subset G$ is a discrete subgroup.*

Hermitian Lie groups

We will consider now the question of whether a left invariant Hermitian structure on a Lie group may have a *flat* first canonical Hermitian connection.

For instance, can the first canonical connection of a left invariant Hermitian structure coincide with the $(-)$-connection?

Theorem

Let $(\mathfrak{g}, J, g)$ be a Hermitian Lie algebra such that its associated first canonical connection ∇^1 satisfies $\nabla^1_x y = 0$ for every $x, y \in \mathfrak{g}$, that is, ∇^1 coincides with the $(-)$-connection. Then

- *J is abelian,*
- *$\mathfrak{g}$ is abelian.*

A similar result for the Chern connection does not hold, according to [Di Scala-Vezzoni '11]

Motivated by previous results, we study next Hermitian structures on Lie algebras whose associated first canonical connection is flat, when the complex structure is abelian.

Lemma

Let $(\mathfrak{g}, J, g)$ be a Hermitian Lie algebra with J abelian. If the associated first canonical connection ∇^1 is flat, then $\mathfrak{z} \cap \mathfrak{g}' = \{0\}$.

Straightforward consequence:

Proposition

Let $(\mathfrak{g}, J, g)$ be a Hermitian Lie algebra with $\mathfrak{g}$ nilpotent and J abelian. If the associated first canonical connection ∇^1 is flat, then $\mathfrak{g}$ is abelian.

Corollary

Let $N = \Gamma \backslash G$ be a nilmanifold with a left invariant Hermitian structure (J, g) such that J is abelian. If the associated first canonical connection ∇^1 is flat, then N is diffeomorphic to a torus.

However, with some extra effort, the hypothesis of nilpotency can be dropped. Indeed, we have

Theorem

Let $(\mathfrak{g}, J, g)$ be a Hermitian Lie algebra with J abelian. If the associated first canonical connection ∇^1 is flat, then $\mathfrak{g}$ is abelian.

Idea of the proof: show that ∇^1 coincides with the $(-)$-connection, that is, $\nabla^1 \equiv 0$. By a previous result, $\mathfrak{g}$ is abelian.

Remark

The hypothesis of abelian complex structure cannot be omitted. Indeed, any even-dimensional Lie group with a flat left invariant metric admits a complex structure which makes it Kähler (Barberis-Dotti-Fino '06). In this case, $\nabla^1 = \nabla^g$ is flat.