On the geometry of $\mathcal{PR}$-warped products in para-Kähler manifolds

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(based on a joint work with Bang Yen Chen, MSU, USA)

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Outline

1. **CR-submanifolds**
   - Basic Properties

2. **CR-products in Kähler manifolds**
   - **CR-products**
   - Warped product **CR-submanifolds** in Kähler manifolds

3. Contact **CR-products** in Sasakian manifolds
   - Contact **CR-products**
   - Contact **CR** warped products

4. **PR warped products** in para-Kähler manifolds
   - **PR** products
   - **PR** warped products
... from the beginning

\[(M, g) \leftrightarrow (\tilde{M}, \tilde{g}, J) \text{ – Kähler manifold}\]

\[T(M) \text{ its tangent bundle; } T(M) \perp \text{ its normal bundle}\]

Two important situations occur:

- If \(J(T_x M) = T_x M\) for all \(x \in M\), \(M\) is called a complex submanifold or holomorphic submanifold.
- If \(J(T_x M) \subset T(M) \perp x\) for all \(x \in M\), \(M\) is known as a totally real submanifold.
... from the beginning

\((M, g) \leftrightarrow (\tilde{M}, \tilde{g}, J)\) – Kähler manifold

\(T(M)\) its tangent bundle; \(T(M)^\perp\) its normal bundle

Two important situations occur:

- \(T_x(M)\) is invariant under the action of \(J\):
  \[
  J(T_x(M)) = T_x(M) \quad \text{for all } x \in M
  \]

\(M\) is called \textit{complex} submanifold or \textit{holomorphic} submanifold
... from the beginning

\[(M, g) \leftrightarrow \left( \tilde{M}, \tilde{g}, J \right) \text{ – Kähler manifold} \]

\[T(M) \text{ its tangent bundle; } T(M) \perp \text{ its normal bundle} \]

Two important situations occur:

- \(T_x(M)\) is anti-invariant under the action of \(J\):

\[J(T_x(M)) \subset T(M)_x \perp \text{ for all } x \in M\]

\(M\) is known as a **totally real** submanifold
... from the beginning

In 1978 A. Bejancu

- *CR-submanifolds of a Kähler manifold. I*,
- *CR-submanifolds of a Kähler manifold. II*,

started a study of the geometry of a class of submanifolds situated between the two classes mentioned above.

Such submanifolds were named *CR–submanifolds*: 
In 1978 A. Bejancu

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- CR-submanifolds of a Kähler manifold. II,

started a study of the geometry of a class of submanifolds situated between the two classes mentioned above.

Such submanifolds were named CR–submanifolds:

\[ M \text{ is a CR-submanifold of a Kähler manifold } (\tilde{M}, \tilde{g}, J) \text{ if there exists a holomorphic distribution } D \text{ on } M, \text{ i.e. } JD_x = D_x, \forall x \in M \text{ and such that its orthogonal complement } D_x^\bot \text{ is anti-invariant, namely } JD_x^\bot \subset T(M)_x^\bot, \forall x \in M. \]
Notations

For any $X$ tangent to $M$:

\[ PX = \tan(JX) \text{ and } FX = \nor(JX) \]

For any $N$ normal to $M$:

\[ tN = \tan(JN) \text{ and } fN = \nor(JN) \]

Here $\tan$ and $\nor$ denotes the tangential and respectively the normal component.
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Here $\tan$ and $\nor$ denotes the tangential and respectively the normal component.

Denote by $\nu$ the complementary orthogonal subbundle:

$$T(M)^\perp = JD^\perp \oplus \nu \quad JD^\perp \perp \nu$$
Submanifold formulas

Gauss and Weingarten formulae

(G) $\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$

(W) $\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N$

for any $X, Y \in \chi(M)$, and $N \in \Gamma^\infty(T(M)^\perp)$.

$\nabla$ is the induced connection
$\nabla^\perp$ is the normal connection
$\sigma$ is the second fundamental form
$A_N$ is the Weingarten operator

$$g(A_N X, Y) = \tilde{g}(N, \sigma(X, Y))$$
Integrability

Proposition (Bejancu - 1979, Blair & Chen - 1979)

The totally real distribution $\mathcal{D} \perp$ of a CR-submanifold in a Kähler manifold is always integrable.

Proposition (Blair & Chen - 1979)

The distribution $\mathcal{D}$ is integrable if and only if

$$\tilde{g}(\sigma(X, JY), JZ) = \tilde{g}(\sigma(JX, Y), JZ), \quad \forall \, X, Y \in \mathcal{D} \text{ and } Z \in \mathcal{D} \perp.$$ 

Proposition (Bejancu, Kon & Yano - 1981)

For a CR-submanifold $M$ in a Kähler manifold, the leaf $\mathcal{N} \perp$ of $\mathcal{D} \perp$ is totally geodesic in $M$ if and only if

$$\tilde{g}(\sigma(\mathcal{D}, \mathcal{D} \perp), J\mathcal{D} \perp) = 0.$$
Every $CR$-submanifold of a Kähler manifold is foliated by totally real submanifolds.
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**Definition (Chen - 1981)**

A $CR$-submanifold of a Kähler manifold $\tilde{M}$ is called $CR$-product if it is locally a Riemannian product of a holomorphic submanifold $N^\top$ and a totally real submanifold $N^\perp$ of $\tilde{M}$. 
Theorems of characterization

Theorem (Chen - 1981)

A \( CR \)-submanifold of a Kähler manifold is a \( CR \)-product if and only if \( P \) is parallel.
Theorems of characterization

**Theorem (Chen - 1981)**

A CR-submanifold of a Kähler manifold is a CR-product if and only if \( P \) is parallel.

**Proof.**

\( N^\top \) is a leaf of \( D \)

\( N^\top \) and \( N^\perp \) are totally geodesic in \( M \)
Every CR-product $M$ in $\mathbb{C}^m$ is locally the Riemannian product of a holomorphic submanifold in a linear complex subspace $C^k$ and a totally real submanifold of a $C^{m-k}$, i.e.

$$M = N^\top \times N^\perp \subset \mathbb{C}^k \times \mathbb{C}^{m-k}.$$
Segre embedding:

\[ S_{sq} : \mathbb{C}P^s \times \mathbb{C}P^q \longrightarrow \mathbb{C}P^{s+q+sq} \]

\((z_0, \ldots, z_s; w_0, \ldots, w_q) \mapsto (z_0 w_0, \ldots, z_i w_j, \ldots, z_s w_q)\)

\(N^\perp = q\)-dimensional totally real submanifold in \(\mathbb{C}P^q\)

\(\mathbb{C}P^s \times N^\perp\) induces a natural \(CR\)-product in \(\mathbb{C}P^{s+q+sq}\) via \(S_{sq}\)
**CR-products in \( \mathbb{C}P^m \)**

Segre embedding:

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\[ (z_0, \ldots, z_s; w_0, \ldots, w_q) \mapsto (z_0w_0, \ldots, z_iw_j, \ldots, z_sw_q) \]

\( N^\perp = q \)-dimensional totally real submanifold in \( \mathbb{C}P^q \)

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**Remark (Chen - 1981)**

There exists no proper CR-product in any complex hyperbolic space \( \tilde{M}(c) \), \( (c < 0) \).
Length of the second fundamental form

Theorem (Chen - 1981)

Let $M$ be a CR-product in $\mathbb{C}P^m$. Then we have

$$||\sigma||^2 \geq 4sq.$$  

If the equality sign holds, then $N^\top$ and $N^\perp$ are both totally geodesic in $\mathbb{C}P^m$. Moreover, the immersion is rigid*. In this case $N^\top$ is a complex space form of constant holomorphic sectional curvature 4, and $N^\perp$ is a real space form of constant sectional curvature 1.

* the Riemannian structure on the submanifold $M$ is completely determined as well as the second fundamental form and the normal connection
Warped Products $N^\perp \times_f N^\top$

$(B, g_B), (F, g_F)$ Riemannian manifolds, $f > 0$ smooth function on $B$
$M = B \times_f F$, $g = g_B + f^2 g_F$
Warped Products $N^\perp \times_f N^\top$

$(B, g_B), (F, g_F)$ Riemannian manifolds, $f > 0$ smooth function on $B$

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**Theorem (Chen - 2001)**

If $M = N^\perp \times_f N^\top$ is a warped product $CR$-submanifold of a Kähler manifold $\tilde{M}$ such that $N^\perp$ is a totally real submanifold and $N^\top$ is a holomorphic submanifold of $\tilde{M}$, then $M$ is a $CR$-product.
**Warped Products** $N^{\perp} \times_f N^{^\top}$

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**Proof.**

$f$ should be a constant and $A_{J^D \perp D} = 0$ is verified.
Warped Products $N^\perp \times_f N^\top$

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**Proof.**

$f$ should be a constant and $A_{JD\perp D} = 0$ is verified.

**Remark (Chen - 2001)**

There do not exist warped product $CR$-submanifolds of the form $N^\perp \times_f N^\top$ other than $CR$-products.
Warped Products $N^\top \times_f N^\perp$

By contrast, there exist many warped product $CR$-submanifolds $N^\top \times_f N^\perp$ which are not $CR$-products.
Warped Products $N^\top \times_f N^\perp$

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$\downarrow$

$CR$-warped products
Warped Products $N^\top \times_f N^\bot$

By contrast, there exist many warped product CR-submanifolds $N^\top \times_f N^\bot$ which are not CR-products.

\[ \downarrow \]

CR-warped products

**Theorem (Chen - 2001)**

A proper CR-submanifold $M$ of a Kähler manifold $\tilde{M}$ is locally a CR-warped product if and only if

\[ A_{JZ}X = ((JX)\mu)Z, \quad X \in D, \quad Z \in D^\perp \]

for some function $\mu$ on $M$ satisfying $W\mu = 0$, for all $W \in D^\perp$. 

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Geometry of PR-warped products

Golden Sands, September '11
A general Inequality for CR-warped products

**Theorem (Chen - 2001)**

Let $M = N^\top \times_f N^\perp$ be a CR-warped product in a Kähler manifold $\tilde{M}$. Then

1. $||\sigma||^2 \geq 2q||\nabla(\log f)||^2$, where $\nabla(\log f)$ is the gradient of $\log f$

2. If the equality sign holds identically, then $N^\top$ is a totally geodesic and $N^\perp$ is a totally umbilical submanifold of $\tilde{M}$. Moreover, $M$ is a minimal submanifold in $\tilde{M}$

3. When $M$ is generic and $q > 1$, the equality sign holds if and only if $N^\perp$ is a totally umbilical submanifold of $\tilde{M}$

4. When $M$ is generic and $q = 1$, then the equality sign holds if and only if the characteristic vector of $M$ is a principal vector field with zero as its principal curvature.
   (In this case $M$ is a real hypersurface in $\tilde{M}$.)
Equality sign when \( \tilde{M} = \tilde{M}(c) \)

For \( CR \)-warped products in complex space forms:

**Theorem (Chen - 2001)**

Let \( M = N^\top \times_f N^\perp \) be a non-trivial \( CR \)-warped product in a complex space form \( \tilde{M}(c) \), satisfying \( ||\sigma||^2 = 2q||\nabla(\log f)||^2 \). Then

1. \( N^\top \) is a totally geodesic holomorphic submanifold of \( \tilde{M}(c) \). Hence \( N^\top \) is a complex space form \( N^s(c) \) of constant holomorphic sectional curvature \( c \)

2. \( N^\perp \) is a totally umbilical totally real submanifold of \( \tilde{M}(c) \). Hence, \( N^\perp \) is a real space form of constant sectional curvature, say \( \epsilon > c/4 \)
Equality sign when $\tilde{M} = \mathbb{C}^m$

**Theorem (Chen - 2001)**

A CR-warped product $M = N^\top \times_f N^\perp$ in a complex Euclidean $m$-space $\mathbb{C}^m$ satisfies the equality if and only if

1. $N^\top$ is an open portion of a complex Euclidean $s$ space $\mathbb{C}^s$
2. $N^\perp$ is an open portion of the unit $q$-sphere $S^q$
3. up to a rigid motion of $\mathbb{C}^m$, the immersion of $M \subset \mathbb{C}^s \times_f S^q$ into $\mathbb{C}^m$ is

$$r(z, w) = (z_1 + (w_0 - 1)a_1 \sum_{j=1}^{n} a_j z_j, \ldots, z_s + (w_0 - 1)a_s \sum_{j=1}^{n} a_j z_j,\ldots, 0)$$

$$w_1 \sum_{j=1}^{n} a_j z_j, \ldots, w_q \sum_{j=1}^{n} a_j z_j, 0, \ldots, 0)$$

$$z = (z_1, \ldots, z_s) \in \mathbb{C}^s, \ w = (w_0, \ldots, w_q) \in S^q \in \mathbb{B}^{q+1}$$

$$f = \sqrt{<a, z>^2 + <ia, z>^2}, \text{ for some point } a = (a_1, \ldots, a_s) \in S^{s-1} \in \mathbb{B}^s.$$
Sasakian manifolds

Another line of thought, similar to that concerning Sasakian geometry as an odd dimensional version of Kählerian geometry, led to the concept of a contact $CR$-submanifold:

$(\tilde{M}^{2m+1}, \phi, \xi, \eta, \tilde{g})$ Sasakian manifold: $\phi \in T^{1}_1(\tilde{M}), \xi \in \chi(\tilde{M}), \eta \in \Lambda^1(\tilde{M})$:

$\phi^2 = -I + \eta \otimes \xi, \phi \xi = 0, \eta \circ \phi = 0, \eta(\xi) = 1$

$d\eta(X, Y) = \tilde{g}(X, \phi Y)$ \hspace{1cm} (the contact condition)

$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y)$ \hspace{1cm} (the compatibility condition)

$(\tilde{\nabla}_U \phi) V = -\tilde{g}(U, V)\xi + \eta(V)U, \hspace{0.5cm} U, V \in \chi(\tilde{M})$
A contact $CR$ submanifold $M$ of a Sasakian manifold $\tilde{M}$ is called **contact $CR$ product** if it is locally a Riemannian product of a $\phi$-invariant submanifold $N^\top$ tangent to $\xi$ and a totally real submanifold $N^\perp$ of $\tilde{M}$, i.e. $N^\perp$ is $\phi$ anti-invariant submanifold of $\tilde{M}$. 
Contact CR-products

A contact CR submanifold $M$ of a Sasakian manifold $\tilde{M}$ is called **contact CR product** if it is locally a Riemannian product of a $\phi$-invariant submanifold $N^\top$ tangent to $\xi$ and a totally real submanifold $N^\bot$ of $\tilde{M}$, i.e. $N^\bot$ is $\phi$ anti-invariant submanifold of $\tilde{M}$.

**Theorem (M. - 2005)**

Let $M$ be a contact CR submanifold of a Sasakian manifold $\tilde{M}$, $\xi \in \mathcal{D}$. Then $M$ is a contact CR product if and only if $P$ satisfies

$$(\nabla_U P) V = -g(U_\mathcal{D}, V)\xi + \eta(V) U_\mathcal{D}$$

for all $U, V$ tangent to $M$ where $U_\mathcal{D}$ is the $\mathcal{D}$-component of $U$. 
Theorem (M. - 2005)

Let $M$ be a complete, generic, simply connected contact $CR$ submanifold of a complete, simply connected Sasakian space form $\tilde{M}^{2m+1}(c)$.

If $M$ is a contact $CR$ product then

1. either $c \neq -3$ and $M$ is a $\phi$ anti-invariant submanifold of $\tilde{M}$ case in which $M$ is locally a Riemannian product of an integral curve of $\xi$ and a totally real submanifold $N^{\perp}$ of $\tilde{M}$,

2. or $c = -3$ and $M$ is locally a Riemannian product of $\mathbb{R}^{2s+1}$ and $N^{\perp}$ where $\mathbb{R}^{2s+1}$ is endowed with the usual Sasakian structure and $N^{\perp}$ is a totally real submanifold of $\mathbb{R}^{2m+1}$ (with the usual Sasakian structure).
Theorem (M. - 2005)

Let $\tilde{M}$ be a Sasakian manifold and let $M = N^\perp \times_f N^\top$ be a warped product $CR$ submanifold such that $N^\perp$ is a totally real submanifold and $N^\top$ is $\phi$ holomorphic (invariant) of $\tilde{M}$. Then $M$ is a $CR$ product.
Characterization theorem

**Theorem (M. - 2005)**

Let $\tilde{M}$ be a Sasakian manifold and let $M = N^\perp \times_f N^\top$ be a warped product CR submanifold such that $N^\perp$ is a totally real submanifold and $N^\top$ is $\phi$ holomorphic (invariant) of $\tilde{M}$. Then $M$ is a CR product.

A contact CR submanifold $M$ of a Sasakian manifold $\tilde{M}$, tangent to $\xi$ is called a contact CR warped product if it is the warped product $N^T \times_f N^\perp$ of an invariant submanifold $N^T$, tangent to $\xi$ and a tot. real submanifold $N^\perp$ of $\tilde{M}$. 
Characterization theorem

Theorem (M. - 2005)

Let \( \tilde{M} \) be a Sasakian manifold and let \( M = N^\perp \times_f N^\top \) be a warped product \( CR \) submanifold such that \( N^\perp \) is a totally real submanifold and \( N^\top \) is \( \phi \) holomorphic (invariant) of \( \tilde{M} \). Then \( M \) is a \( CR \) product.

A contact \( CR \) submanifold \( M \) of a Sasakian manifold \( \tilde{M} \), tangent to \( \xi \) is called a contact \( CR \) warped product if it is the warped product \( N^T \times_f N^\perp \) of an invariant submanifold \( N^T \), tangent to \( \xi \) and a tot. real submanifold \( N^\perp \) of \( \tilde{M} \).

Theorem (M. - 2005)

A strictly proper \( CR \) submanifold \( M \) of a Sasakian manifold \( \tilde{M} \), tangent to \( \xi \), is locally a contact \( CR \) warped product if and only if there exists \( \mu \in C^\infty(M) \) satisfying \( W\mu = 0 \) for all \( W \in D^\perp \).

\[
A_{\phi}ZX = (\eta(X) - (\phi X)(\mu)) \ Z , \quad X \in D , \quad Z \in D^\perp .
\]
A general inequality

(the ambient $\tilde{M}$ is not necessary a Sasakian space form)

**Theorem (Hasegawa & I. Mihai - 2003, M. - 2005)**

Let $M = N^\top \times_f N^\perp$ be a contact $CR$ warped product in $\tilde{M}$. We have

1. $||\sigma||^2 \geq 2q (||\nabla \ln f||^2 + 1)$
2. If the equality sign holds, then $N^\top$ is a totally geodesic submanifold and $N^\perp$ is a totally umbilical submanifold of $\tilde{M}$. The product manifold $M$ is a minimal submanifold in $\tilde{M}$. 

(Notice that $M$ is a hypersurface in $\tilde{M}$ with the unitary normal vector $\mu$).
A general inequality

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1. $||\sigma||^2 \geq 2q (||\nabla \ln f||^2 + 1)$
2. If the equality sign holds, then $N^\top$ is a totally geodesic submanifold and $N^\perp$ is a totally umbilical submanifold of $\tilde{M}$. The product manifold $M$ is a minimal submanifold in $\tilde{M}$.

3. The case $TM^\perp = \phi D^\perp$. If $q > 1$ then the equality sign holds identically if and only if $N^\perp$ is a totally umbilical submanifold of $\tilde{M}$.

4. If $q = 1$ then the equality sign holds identically if and only if the characteristic vector field $\phi \mu$ of $M$ satisfies $A_\mu \phi \mu = -\phi \nabla \ln f - \xi$.

(Notice that $M$ is a hypersurface in $\tilde{M}$ with the unitary normal vector $\mu$).
A good geometric inequality

Theorem (I. Mihai - 2004, M. - 2005)

Let $M = N^\top \times_f N^\perp$ be a contact CR warped product of a Sasakian space form $\tilde{M}^{2m+1}(c)$. Then

$$||\sigma||^2 \geq 2q \left[ ||\nabla \ln f||^2 - \Delta \ln f + \frac{c + 3}{2} s + 1 \right].$$
A good geometric inequality

**Theorem (I. Mihai - 2004, M. - 2005)**

Let $M = N^T \times_f N^\perp$ be a contact CR warped product of a Sasakian space form $\tilde{M}^{2m+1}(c)$. Then

$$||\sigma||^2 \geq 2q \left[ ||\nabla \ln f||^2 - \Delta \ln f + \frac{c + 3}{2} s + 1 \right].$$

**Proof.**

$$||\sigma(D, D^\perp)||^2 = \sum_{j=1}^{2s+1} \sum_{\alpha=1}^{q} ||\sigma(X_j, Z_\alpha)||^2$$

$$||\sigma_{\phiD^\perp}(D, D^\perp)||^2 = \sum_{\alpha=1}^{q} ||\nabla \ln f||^2 + \sum_{\alpha=1}^{q} ||\phi Z_\alpha||^2$$

$$2 \sum_{j=1}^{s} \sum_{\alpha=1}^{q} \left\{ ||\sigma_\nu(e_j, Z_\alpha)||^2 + ||\sigma_\nu(\phi e_j, Z_\alpha)||^2 \right\} = (c + 3)sq - 2q\Delta(\ln f).$$
An almost para-Hermitian manifold is a manifold $\tilde{M}$ equipped with an almost product structure $\mathcal{P} \neq \pm I$ and a pseudo-Riemannian metric $\tilde{g}$ such that

$$\mathcal{P}^2 = I, \quad \tilde{g}(\mathcal{P}X, \mathcal{P}Y) = -\tilde{g}(X, Y).$$

An almost para-Hermitian manifold is called para-Kähler if it satisfies $\tilde{\nabla}\mathcal{P} = 0$ identically.

A pseudo-Riemannian submanifold $M$ of a para-Kähler manifold $\tilde{M}$ is called invariant if the tangent bundle of $M$ is invariant under the action of $\mathcal{P}$.

$M$ is called anti-invariant if $\mathcal{P}$ maps each tangent space $T_p M$, $p \in M$, into the normal space $T_p^\perp M$.

We put $\|X\|^2 = \tilde{g}(X, X)$. 
PR products

A pseudo-Riemannian submanifold $M$ of a para-Kähler manifold $\tilde{M}$ is called a **PR-submanifold** if the tangent bundle $TM$ of $M$ is the direct sum of an *invariant* distribution $D$ and an *anti-invariant* distribution $D^\perp$:

$$T(M) = D \oplus D^\perp, \quad PD = D, \quad PD^\perp \subseteq T_p^\perp(M).$$

A **PR-product** of a para-Kähler manifold is called a **PR-product** if it is locally a direct product $N_T \times N_\perp$ of an invariant submanifold $N_T$ and an anti-invariant submanifold $N_\perp$. 
Proposition (Chen, M. - to appear)

A \( PR \)-submanifold of a para-Kähler manifold is a \( PR \)-product if and only if \( P \) is parallel.

Proposition (Chen - 2011)

Let \( N_{\top} \times N_{\bot} \) be a \( PR \)-product of the para-Kähler \((h + p)\)-plane \( P^{h+p} \) with \( h = \frac{1}{2} \dim N_{\top} \) and \( p = \dim N_{\bot} \). If \( N_{\bot} \) is either spacelike or timelike, then the \( PR \)-product is an open part of a direct product of a para-Kähler \( h \)-plane \( P^{h} \) and a Lagrangian submanifold \( L \) of \( P^{p} \), i.e.,

\[
N_{\top} \times N_{\bot} \subset P^{h} \times L \subset P^{h} \times P^{p} = P^{h+p}.
\]
If a \( \mathcal{P}R \)-submanifold \( M \) is a warped product \( N_\perp \times f N_\top \) of an anti-invariant submanifold \( N_\perp \) and an invariant submanifold \( N_\top \) with warping function \( f : N_\perp \rightarrow \mathbb{R}_+ \), then \( M \) is a \( \mathcal{P}R \) product \( N_\perp \times N^f_\top \), where \( N^f_\top \) is the manifold \( N_\top \) endowed with the homothetic metric \( g^f_\top = f^2 g_\top \).
**Proposition (Chen, M.)**

If a $\mathcal{P}R$-submanifold $M$ is a warped product $N_{\perp} \times f N_{\top}$ of an anti-invariant submanifold $N_{\perp}$ and an invariant submanifold $N_{\top}$ with warping function $f : N_{\perp} \rightarrow \mathbb{R}_{+}$, then $M$ is a $\mathcal{P}R$ product $N_{\perp} \times N_{\top}^{f}$, where $N_{\top}^{f}$ is the manifold $N_{\top}$ endowed with the homothetic metric $g_{\top}^{f} = f^{2} g_{\top}$.

A $\mathcal{P}R$-submanifold of a para-Kähler manifold $\tilde{M}$ is called a $\mathcal{P}R$-warped product if it is a warped product of the form: $N_{\top} \times f N_{\perp}$, where $N_{\top}$ in an invariant submanifold, $N_{\perp}$ is an anti-invariant submanifold of $M$ and $f$ is a non-constant function $f : N_{\top} \rightarrow \mathbb{R}_{+}$. 
Proposition (Chen, M.)

Let $M$ be a proper $\mathcal{P}R$-submanifold of a para-Kähler manifold. Then $M$ is a $\mathcal{P}R$-warped product if and only if

$$A_{FZ}X = (PX(\mu))Z,$$

for some smooth function $\mu$ on $M$ satisfying $W(\mu) = 0$, $\forall W \in \mathcal{D}^\perp$.
An optimal inequality

**Theorem (Chen, M.)**

Let \( M = N_\top \times_f N_\bot \) be a \( PR \)-warped product in a para-Kähler manifold \( \tilde{M} \). Suppose that \( N_\bot \) is space-like and \( \nabla_\bot (\mathcal{P} N_\bot) \subseteq \mathcal{P} N_\bot \). Then the second fundamental form of \( M \) satisfies

\[
S_\sigma \leq 2p \| \nabla \ln f \|_2 + \| \sigma^D_\nu \|_2,
\]

where \( p = \dim N_\bot \), \( S_\sigma = \tilde{g}(\sigma, \sigma) \), \( \nabla \ln f \) is the gradient of \( \ln f \) with respect to the metric \( g \) and \( \| \sigma^D_\nu \|_2 = \tilde{g}(\sigma_\nu(D, D), \sigma_\nu(D, D)) \). Here the index \( \nu \) represents the \( \nu \)-component of that object.
An optimal inequality

Theorem (Chen, M.)

Let $M = N^\top \times_f N_\perp$ be a $\mathcal{P}R$-warped product in a para-Kähler manifold $\tilde{M}$. Suppose that $N_\perp$ is space-like and $\nabla_\perp(\mathcal{P}N_\perp) \subseteq \mathcal{P}N_\perp$. Then the second fundamental form of $M$ satisfies

$$S_\sigma \leq 2p \|\nabla \ln f\|_2 + \|\sigma_\nu^D\|_2,$$

where $p = \dim N_\perp$, $S_\sigma = \tilde{g}(\sigma, \sigma)$, $\nabla \ln f$ is the gradient of $\ln f$ with respect to the metric $g$ and $\|\sigma_\nu^D\|_2 = \tilde{g}(\sigma_\nu(D, D), \sigma_\nu(D, D))$. Here the index $\nu$ represents the $\nu$-component of that object.

Remark (Chen, M.)

If the manifold $N_\perp$ in the previous Theorem is time-like, then

$$S_\sigma \geq 2p \|\nabla \ln f\|_2 + \|\sigma_\nu^D\|_2.$$
An optimal inequality

Remark (Chen, M.)

For every $\mathcal{P}R$-warped product $N_T \times N_\perp$ in a para-Kähler manifold $\tilde{M}$, $\dim \tilde{M} \geq \dim N_T + 2 \dim N_\perp$ holds. Thus the smallest codimension is $\dim N_\perp$. 
An optimal inequality

Remark (Chen, M.)

For every $\mathcal{PR}$-warped product $N_T \times N_\perp$ in a para-Kähler manifold $\tilde{M}$, $\dim \tilde{M} \geq \dim N_T + 2 \dim N_\perp$ holds. Thus the smallest codimension is $\dim N_\perp$.

Theorem (Chen, M.)

Let $N_T \times_f N_\perp$ be a $\mathcal{PR}$-warped product in a para-Kähler manifold $\tilde{M}$. If $N_\perp$ is space-like (respectively, time-like) and $\dim \tilde{M} = \dim N_T + 2 \dim N_\perp$, then the second fundamental form of $M$ satisfies

$$S_\sigma \leq 2p\|\nabla \ln f\|_2 \quad \text{(respectively, } S_\sigma \geq 2p\|\nabla \ln f\|_2).$$

If the equality sign holds identically, we have

$$\sigma(D, D) = \sigma(D_\perp, D_\perp) = \{0\}.$$
Main Theorem

Theorem (Chen, M.)

Let \( N_\top \times_f N_\bot \) be a space-like \( PR \)-warped product in the para-Kähler \((h + p)\)-plane \( P^{h+p} \) with \( h = \frac{1}{2} \dim N_\top \) and \( p = \dim N_\bot \). Then we have

\[
S_\sigma \leq 2p \| \nabla \ln f \|_2.
\]

The equality sign holds identically if and only if \( N_\top \) is an open part of a para-Kähler \( h \)-plane, \( N_\bot \) is an open part of \( S^p, E^p \) or \( H^p \), and the immersion is given by one of the following:
1. \( \Phi : D_1 \times_f S^p \rightarrow P^{h+p}; \)

\[
\Phi(z, w) = \left( z_1 + \bar{v}_1(w_0 - 1) \sum_{j=1}^{h} v_j z_j, \ldots, z_h + \bar{v}_h(w_0 - 1) \sum_{j=1}^{h} v_j z_j, w_1 \sum_{j=1}^{h} j v_j z_j, \ldots, w_p \sum_{j=1}^{h} j v_j z_j \right), \quad h \geq 2,
\]

with warping function

\[
f = \sqrt{\langle \bar{v}, z \rangle^2 - \langle j \bar{v}, z \rangle^2},
\]

where \( v = (v_1, \ldots, v_h) \in S^{2h-1} \subseteq \mathbb{D}^h, w = (w_0, w_1, \ldots, w_p) \in S^p, z = (z_1, \ldots, z_h) \in D_1 \) and \( D_1 = \{ z \in \mathbb{D}^h : \langle \bar{v}, z \rangle^2 > \langle j \bar{v}, z \rangle^2 \}. \)
2.

\[ \Phi : D_1 \times f \mathbb{H}^p \rightarrow \mathcal{P}^{h+p}; \]

\[ \Phi(z, w) = \left( z_1 + \bar{v}_1(w_0 - 1) \sum_{j=1}^{h} v_j z_j, \ldots, z_h + \bar{v}_h(w_0 - 1) \sum_{j=1}^{h} v_j z_j, \right. \]

\[ \left. w_1 \sum_{j=1}^{h} j v_j z_j, \ldots, w_p \sum_{j=1}^{h} j v_j z_j \right), \quad h \geq 1, \]

with the warping function

\[ f = \sqrt{\langle \bar{v}, z \rangle^2 - \langle j \bar{v}, z \rangle^2}, \]

where \( v = (v_1, \ldots, v_h) \in \mathbb{H}^{2h-1} \subset \mathbb{D}^h \), \( w = (w_0, w_1, \ldots, w_p) \in \mathbb{H}^p \) and \( z = (z_1, \ldots, z_h) \in D_1 \).
3. \( \Phi(z, u) : D_1 \times_f \mathbb{E}^p \longrightarrow \mathcal{P}^{h+p}; \)

\[
\Phi(z, u) = \left( z_1 + \frac{\bar{v}_1}{2} \left( \sum_{a=1}^{p} u_a^2 \right) \sum_{j=1}^{h} v_j z_j, \ldots, z_h + \frac{\bar{v}_h}{2} \left( \sum_{a=1}^{p} u_a^2 \right) \sum_{j=1}^{h} v_j z_j, u_1 \sum_{j=1}^{h} j v_j z_j, \ldots, u_p \sum_{j=1}^{h} j v_j z_j \right), \quad h \geq 2,
\]

with the warping function

\[
f = \sqrt{\langle \bar{v}, z \rangle^2 - \langle j \bar{v}, z \rangle^2},
\]

where \( v = (v_1, \ldots, v_h) \) is a light-like vector in \( \mathbb{D}^h, z = (z_1, \ldots, z_h) \in D_1 \) and \( u = (u_1, \ldots, u_p) \in \mathbb{E}^p, \)

Moreover, in this case, each leaf \( \mathbb{E}^p \) is quasi-minimal in \( \mathcal{P}^{h+p}. \)
4.

\[ \Phi(z, u) : D_2 \times_f \mathbb{E}^p \longrightarrow \mathcal{P}^{h+p}; \]

\[ \Phi(z, u) = \left( z_1 + \frac{v_1}{2} \sum_{a=1}^{p} u_a^2, \ldots, z_h + \frac{v_h}{2} \sum_{a=1}^{p} u_a^2, \frac{v_0}{2} u_1, \ldots, \frac{v_0}{2} u_p \right), \ h \geq 1, \]

with the warping function

\[ f = \sqrt{-\langle v, z \rangle}, \]

where \( v_0 = \sqrt{b_1} + \epsilon \sqrt{b_1} \) with \( b_1 > 0 \), \( D_2 = \{ z \in \mathbb{D}^h : \langle v, z \rangle < 0 \} \), \( v = (v_1, \ldots, v_h) = (b_1 + \epsilon_j b_1, \ldots, b_h + \epsilon_j b_h), \) \( \epsilon = \pm 1 \), 
\( z = (z_1, \ldots, z_h) \in D_2 \) and \( u = (u_1, \ldots, u_p) \in \mathbb{E}^p. \)

In each of the four cases the warped product is minimal in \( \mathbb{E}^{2(h+p)} \).
Sketch of Proof

Since $N_{\perp}$ is a space-like totally umbilical, non-totally geodesic submanifold in $\mathcal{P}^m$, it is congruent

- either to the Euclidean $p$-sphere $\mathbb{S}^p$,
- or to the hyperbolic $p$-plane $\mathbb{H}^p$,
- or to a flat quasi-minimal submanifold $\mathbb{E}^p$. 
Sketch of Proof

The non-constant solutions $\psi = \psi(s_1, \ldots, s_h, t_1, \ldots, t_h)$ of the following system of partial differential equations

\[
\frac{\partial^2 \psi}{\partial s_i \partial s_j} + \frac{\partial \psi}{\partial s_i} \frac{\partial \psi}{\partial s_j} + \frac{\partial \psi}{\partial t_i} \frac{\partial \psi}{\partial t_j} = 0
\]

\[
\frac{\partial^2 \psi}{\partial s_i \partial t_j} + \frac{\partial \psi}{\partial s_i} \frac{\partial \psi}{\partial t_j} + \frac{\partial \psi}{\partial t_i} \frac{\partial \psi}{\partial s_j} = 0
\]

\[
\frac{\partial^2 \psi}{\partial t_i \partial t_j} + \frac{\partial \psi}{\partial t_i} \frac{\partial \psi}{\partial t_j} + \frac{\partial \psi}{\partial s_i} \frac{\partial \psi}{\partial s_j} = 0
\]
Sketch of Proof

are either given by

$$\psi = \frac{1}{2} \ln \left| \left( (\langle \mathbf{v}, z \rangle + c_1)^2 - (\langle j\mathbf{v}, z \rangle + c_2)^2 \right) \right|,$$

where $z = (s_1, s_2, \ldots, s_h, t_1, t_2, \ldots, t_h)$, $\mathbf{v} = (a_1, a_2, \ldots, a_h, 0, b_2, \ldots, b_h)$ is a constant vector in $\mathbb{R}^{2h}$ with $a_1 \neq 0$, $c_1, c_2 \in \mathbb{R}$ and $j\mathbf{v} = (0, b_2, \ldots, b_h, a_1, a_2, \ldots, a_h)$; or given by

$$\psi = \frac{1}{2} \ln \left| (\langle \mathbf{v}_1, z \rangle + c) (\langle \mathbf{v}_2, z \rangle + d) \right|,$$

where $\mathbf{v}_1 = (0, a_2, \ldots, a_h, 0, \epsilon a_2, \ldots, \epsilon a_h)$, $\mathbf{v}_2 = (b_1, \ldots, b_h, -\epsilon b_1, \ldots, -\epsilon b_h)$ with $b_1 \neq 0$, $z$ is as above and $c, d \in \mathbb{R}$.

Here $\langle \ , \ \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^{2h}$. 

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Geometry of $\mathcal{P}R$-warped products

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Thank you for attention!