Geometric Structures in Mathematical Physics

Non-existence of almost complex structures on quaternion-Kähler manifolds of positive type

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Main references:


Definition 1 A Quaternion-Kähler manifold — qK manifold for short — is a (complete) oriented Riemannian manifold $(M, g)$ of dimension $4k \geq 8$, such that the vector bundle $A(M)$ of skew-symmetric endomorphisms of the tangent bundle $TM$ admits a vector subbundle $Q \subset A(M)$ of rank $3$, satisfying the two properties:

(i) $Q$ is preserved by the Levi-Civita connection $\nabla$ of $g$, and

(ii) $Q$ is locally generated by positively oriented almost complex structures $J_1, J_2, J_3$ with $J_1J_2J_3 = -1$. 
Equivalently, the holonomy group of \((M, g)\) is contained in the subgroup \(Sp(1)Sp(k)\) of \(SO(4k)\):

\[
Sp(k) = \text{Hermitian symplectic group} = \text{preserves the Hermitian quaternionic structure of } \mathbb{H}^k = (\mathbb{C}^{2k}, j)
\]

\(Sp(1) = \text{group of quaternions of norm 1.}\)

The inclusion

\[
Sp(k)Sp(1) = Sp(k) \times Sp(1)/\pm 1 \subset SO(4k)
\]

is realized via the map:

\[
Sp(k) \times Sp(1) \to SO(\mathbb{R}^{4k})
\]

\[(A, p) \mapsto \{u \mapsto Aup^{-1}\} \quad \forall u \in \mathbb{H}^k = \mathbb{R}^{4k}.
\]
Model: The quaternionic projective space.

\[ \mathbb{HP}^k = \mathbb{H}^{k+1}/\mathbb{H}^* \] (right action by non-zero quaternions)

For any \( x \) in \( \mathbb{HP}^k \), \( T_x \mathbb{HP}^k = \text{Hom}_{\mathbb{H}}(x, x^\perp) \)

Fix any “\( \mathbb{H} \)-basis” \( u_1 \neq 0 \) in \( x \), \( x \cong \mathbb{H} \) via the map \( p \mapsto u = u_1 p \).

We then set

\[ J_1 X = X \circ_1 i, \quad \text{i.e.} \quad (J_1 X)(u) = X(u_1 ip) \]
\[ J_2 X = X \circ_1 j, \quad \text{i.e.} \quad (J_2 X)(u) = X(u_1 jp) \]
\[ J_3 X = X \circ_1 k, \quad \text{i.e.} \quad (J_3 X)(u) = X(u_1 kp) \]

If \( u_2 = u_1 r \) is any other “\( \mathbb{H} \)-basis” of \( x \), we get

\[ \tilde{J}_1 X = X \circ_2 i = X \circ_1 (rir^{-1}) \]
\[ \tilde{J}_2 X = X \circ_2 j = X \circ_1 (rjr^{-1}) \]
\[ \tilde{J}_3 X = X \circ_2 k = X \circ_1 (rkr^{-1}) \]

These are linear combinations of \( J_1, J_2, J_3 \), hence generate the same rank 3 subbundle of \( A(\mathbb{HP}^k) \).
Other classical examples:

The real Grassmannians $\tilde{Gr}_4(\mathbb{R}^{4+k})$ of oriented 4-dimensional real vector subspaces of $\mathbb{R}^{4+k}$.

The complex Grassmannians $Gr_2(\mathbb{C}^{2+k})$ of 2-dimensional complex vector subspaces of $\mathbb{C}^{2+k}$.

Exercise:

1. Construct a quaternion-Kähler structure $Q$ on these spaces in a similar way, by using the natural identifications:

   $T_x\tilde{Gr}_4(\mathbb{R}^{4+k}) \cong \text{Hom}_\mathbb{R}(x, x^\perp)$

   $T_xGr_2(\mathbb{C}^{2+k}) \cong \text{Hom}_\mathbb{C}(x, x^\perp)$

2. Show that the natural complex structure of $\tilde{Gr}_2(\mathbb{C}^{2+k})$ is not a section of $Q$. 
Basic property:

Quaternion-Kähler manifolds are **Einstein**.

Three types:

**Positive type.** $\text{Scal} > 0$ (M is then compact).

**Type zero.** $\text{Scal} = 0$ : hyperkähler $\Leftrightarrow$ holonomy in $Sp(k) \subset SO(4k)$.

**Negative type.** $\text{Scal} < 0$.

In the sequel of the talk, we only consider qK manifolds of **positive** type.
Definition 2 The **twistor space** of a qK manifold is the sphere bundle \( Z = S(Q) \) of \( Q \), for a suitable norm.

Equivalently, \( Z \) is the 2-sphere bundle over \( M \), whose fiber at each point \( x \) of \( M \) is the set of those complex structures of the tangent space \( T_xM \) which belong to \( Q_x \).

**Basic properties:** (S. Salamon 1982)

1. \( Z \) admits a canonical **integrable** almost complex structure \( J \).

2. The horizontal distribution, \( H^\nabla \), on \( Z \) induced by the Levi-Civita connection \( \nabla \) determines a **holomorphic contact structure** \( \theta \) on \( Z \), with values in the holomorphic line bundle \( L = TM/H^\nabla \):

\[ \theta \wedge (d\theta)^k \] is a nowhere vanishing holomorphic section of \( K_Z \otimes L^{k+1} \), hence determines an isomorphism:

\[ L^{k+1} \cong K_Z^{-1}. \]

3. \( Z \) admits a **Kähler-Eistein metric**, of positive scalar curvature.
It follows that $Z$ is a **Fano contact manifold**, of (complex) dimension $2k + 1$, admitting a Kähler-Einstein metric.

Conversely:

**Theorem 1 (C. LeBrun 1995)** *Any Fano contact manifold admitting a Kähler-Einstein metric — necessarily of positive scalar curvature — is the twistor space of a $q_k$ manifold.*

Alternative proof by A. Morioianu in 1998 in the case when $Z$ is spin (in particular when $k$ is odd. )
Dimension 4:

Definition 3 An oriented (complete) four-dimensional Riemannian manifold $(M, g)$ is called quaternion-Kähler if:

1. $g$ is Einstein, and
2. the conformal class $[g]$ is anti-self-dual, i.e. $W^+ = 0$.

We then have

$$Q \cong A^+ M$$

the bundle of self-dual skew-symmetric endomorphisms, and the twistor space $Z$ coincides with the Atiyah–Hitchin–Singer twistor space.

Examples:

1. The standard sphere $S^4$
2. The complex projective space $\mathbb{CP}^2$, equipped with the Fubini-Study metric and the reverse standard orientation.
The Wolf spaces (J. A. Wolf 1965)

1. Classical Wolf spaces:
\[ \mathbb{H}P^k = Sp(k+1)/Sp(k) \times Sp(1), \]
\[ \tilde{Gr}_4(\mathbb{R}^{4+k}) = SO(4+k)/SO(k) \times SO(4), \]
\[ Gr_2(\mathbb{C}^{2+k}) = SU(2+k)/S(U(k) \times U(1)) \]

2. Exceptional Wolf spaces:
\[ G_2/SO(4), \quad F_4/Sp(3)Sp(1), \]
\[ E_6/SU(6)Sp(1), \quad E_7/Spin(12)Sp(1), \]
\[ E_8/E_7Sp(1) \]

All are symmetric spaces of compact type, one for each compact simple Lie group \( G \).

For each \( G \), the corresponding twistor space has a two-fold description:

1. The \( G \)-adjoint orbit of the highest root \( i\theta \) in \( \mathfrak{g} \) (J. A. Wolf 1965),

2. the projectivization \( \mathbb{P}(O_{\text{min}}) \) of the minimal nilpotent orbit in \( \mathfrak{g}^C = \) the unique closed orbit in \( \mathbb{P}(\mathfrak{g}^C) \) (A. Beauville 1998)
So far, Wolf spaces are the **only known examples** of qK manifolds of positive type.

**Current conjectures:**

$C_1$. All qK manifolds of positive type are Wolf spaces (C. LeBrun–S. Salamon)

$C_2$. All Fano contact manifolds are of the form \( \mathbb{P}(\mathcal{O}_{\text{min}}) \) (A. Beauville)

**Notice:** $C_2$ contains the additional conjecture:

$C_3$: All Fano contact manifolds admit a Kähler-Einstein metric.

**Known results (so far):**

1. The conjecture $C_1$ has been proved if $k = 1$ (N. Hitchin 1981, Friedrich–Kurke 1982) and $k = 2$ (Poon–Salamon 1991, LeBrun–Salamon 1994). As far as I know, $C_1$ has remained open for $k > 2$, and conjectures $C_2, C_3$ for $k \geq 1$.

2. For any $k$, there are only finitely many qK manifolds of positive type (LeBrun-Salamon, *op. cit.*, relying on works of Mori, Višnevski etc... on Fano manifolds).
Compatible almost complex structures on qK manifolds of positive type

An almost complex structure \( J \) on a qK manifold \((M, g, Q)\) is called **compatible** if \( J \) is a section of \( Q \).

The following theorem was established in 1998 by D. Alekseevsky, S. Marchiafava and M. Pontecorvo (with a participation of P. Piccinni):

**Theorem 2** Quaternion-Kähler manifolds of positive type have no globally defined compatible almost complex structure. Equivalently, the twistor fibration \( Z \xrightarrow{\pi} M \) has no global section.

Recall that the natural complex structure of the complex Grassmannians \( Gr_2(\mathbb{C}^{2+k}) \) is **not** compatible with the quaternion-Kähler structure.
The main goal of this talk is to provide a proof of the following theorem:

**Theorem 3** Quaternion-Kähler manifolds of positive type have no globally defined almost complex structure, except for the complex Grassmannians $Gr_2(C^{2+k})$.

**Previously known results:**

1. Quaternionic projective spaces $\mathbb{HP}^k$ have no almost complex structures (F. Hirzebruch 1953, $k \neq 2, 3$, W. S. Massey 1962)

2. Grassmannians $\tilde{Gr}_4(\mathbb{R}^{4+k})$ have no almost complex structure, except, possibly, for $\tilde{Gr}_4(\mathbb{R}^8)$ and $\tilde{Gr}_4(\mathbb{R}^{10})$ (P. Sankaran 1991, Z.-Z. Tang 1994).

3. $S^4$ and $\mathbb{CP}^2$ have no almost complex structure (e.g. use the criterion $\chi + \tau \equiv 0 \mod 4$ for 4-dimensional compact almost complex manifolds).
Proof of Theorem 3.

By the above, we can assume that $k \geq 2$.

The proof then relies on the Atiyah–Singer index theorem and on the following facts:

Fact 1. Any qK manifold $M$ of positive type has $b_2(M) = 0$, except for the complex Grassmannians $Gr_2(\mathbb{C}^{2+k})$ (C. LeBrun–S. Salamon, op. cit.)

We can then assume $b_2(M) = 0$.

Fact 2. For any qK manifold $M$, the complexified tangent bundle $T^C M$ is locally of the form:

$$T^C M = H \otimes \mathbb{C} E$$

where:

$H =$ rank 2 complex vector bundle induced by the standard representation of $Sp(1)$ on $\mathbb{H} = \mathbb{C}^2$,

$E =$ rank $2k$ complex vector bundle induced by the standard representation of $Sp(k)$ on $\mathbb{H}^k = \mathbb{C}^{2k}$. 
Beware however that, except for $\mathbb{HP}^k$, neither $H$ nor $E$ are defined globally, only the quotients $H/\pm 1$ and $E/\pm 1$ are (S. Salamon 1982). Recall that the holonomy group is $Sp(k)Sp(1) = Sp(k) \times Sp(1)/\pm 1$, not $Sp(k) \times Sp(1)$.

**Fact 3.** A qK manifold $M$ of positive type and of dimension $n = 4k$ is spin iff either $M = \mathbb{HP}^k$ or $k$ is even (S. Salamon 1982)

If so, the spinor bundle of $M$ is of the form

$$\Sigma M = \bigoplus_{p+q=k} R^{p,q},$$

by setting

$$R^{p,q} = \text{Sym}^p H \otimes \Lambda^q_0 E$$

where $\text{Sym}^p H$ denotes the $p$-th symmetric complex tensor power of $H$ and $\Lambda^q_0 E$ denotes the primitive part of $\Lambda^q E$ wrt the (complex) symplectic structure of $E$

(cf. *e. g.* Kramer–Semmelmann–Weingart 1999).
The twisted spinor bundle $\Sigma M \otimes R^{p,q}$ is then globally defined — even if $M$ is not spin — iff $p + q + k$ is even.

The corresponding (twisted) Dirac operator is then denoted by $D^{p,q}$.

**Fact 4.** Assume that $p + q + k$ is even, so that $\Sigma M \otimes R^{p,q}$ and $D^{p,q}$ are globally defined on $M$. Then, the index of $D^{p,q}$ is given by

1. $\text{ind}D^{p,q} = 0$ if $p + q < k$
2. $\text{ind}D^{p,q} = (-1)^q (b_{2q-2}(M) + b_{2q}(M))$ if $p + q = k$.

(LeBrun–Salamon *op. cit.*, Semmelmann–Weingart 1982).
Fact 5: Atiyah-Singer Index Theorem.

In general, for any (oriented, even-dimensional) manifold $M$ and for any complex vector bundle $V$ over $M$, the index of the twisted Dirac operator

$$D_V : \Gamma(\Sigma^+ M \otimes V) \to \Gamma(\Sigma^- M \otimes V)$$

is given by the following index formula:

$$\text{ind}D_V = \left( \text{ch}(V) \hat{A}(TM) \right) [M]$$

where $\Sigma^\pm M$ denote the spinor bundles of $M$.

This still makes sense and holds true whenever $\Sigma M \otimes V$ is globally defined, even if $M$ is not spin and $V$ is only defined up to $\pm 1$. 


End of the proof of Theorem 3.

Assume that $M$ is a qK manifold of positive type and apply the index formula to

$$V = T^C M \otimes \text{Sym}^{k-2} H = E \otimes H \otimes \text{Sym}^{k-2} H$$

(recall, we assumed $k \geq 2$).

By using the Clebsch-Gordan decomposition of $H \otimes \text{Sym}^{k-2} H$, we get

$$V = R^{k-1,1} \oplus R^{k-3,1}$$

Notice that $\Sigma M \otimes R^{k-1,1}$ and $\Sigma M \otimes R^{k-3,1}$ are both globally defined — hence also $\Sigma M \otimes V$ — since they both satisfy the rule $p + q + k$ is even (equal to $2k$ and $2k - 2$ respectively).

We then have

$$\text{ind} D_V = \text{ind} D^{k-1,1} + \text{ind} D^{k-3,1}$$

$$= -1 \quad \text{by using Fact 4 with } b_2(M) = 0,$$

$$= \left( \text{ch}(T^C M) \text{ch}(\text{Sym}^{k-2} H) \hat{A}(TM) \right) [M]$$

by using Fact 5 (index formula).
Notice that $\text{ch}(\text{Sym}^{k-2} H)$ is well-defined, even if $\text{Sym}^{k-2} H$ is not (if so, define $\text{ch}(\text{Sym}^{k-2} H)$ as $\left(\text{ch}(\text{Sym}^{k-2} H) \otimes 2\right)^{1/2}$).

Now, assume, for a contradiction, that $M$ admits an almost complex structure, i.e. that $TM = T$ is a complex vector bundle. Then

$$T^C M = T \oplus \overline{T} = T \oplus T^*.$$

Since $H$ is (complex) symplectic, hence self-dual, the term $\text{ch}(\text{Sym}^{k-2} H) \hat{A}(TM)$ in the rhs of the index formula only involves terms of degree $4\ell$.

Since $\text{ch}_j(T^*) = (-1)^j \text{ch}_j(T)$ — components of degree $2j$ — and $M$ is of dimension $4k$, in the above index formula we may replace $T^C$ by $T \oplus T$, hence $\text{ch}(T^C)$ by $2 \text{ch}(T)$. 

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It follows that

\[ \text{ind} D_V = 2 \left( \text{ch} \left( \text{Sym}^{k-2} H \right) \text{ch} (T) \hat{A} (TM) \right) [M]. \]

On the other hand,

\[ \left( \text{ch} \left( \text{Sym}^{k-2} H \right) \text{ch} (T) \hat{A} (TM) \right) [M] \]

is the index of the twisted Dirac operator acting on sections of

\[ \Sigma^+ M \otimes \text{Sym}^{k-2} H \otimes T, \]

which is globally defined on \( M \) (as \( k - 2 + k = 2k - 2 \) is even).

It is then an integer, so that \( \text{ind} D_V \) must be even.

This contradicts \( \text{ind} D_V = -1 \), hence completes the proof of Theorem 3.
Weakly complex qK manifolds:

**Definition 4** A manifold $M$ is weakly complex — or stably complex — if $TM \oplus \mathbb{R}^\ell$ can be given a structure of complex vector bundle, where $\mathbb{R}^\ell$ here stands for the trivial vector bundle of rank $\ell$.

By using an easy variant of the above argument, we get

**Theorem 4** Quaternion-Kähler manifolds of positive type of dimension $4k$, $k \geq 2$, are not weakly complex.

**Proof:** In the last step of the above argument, assume, for a contradiction, that $T \oplus \mathbb{R}^\ell$ is a complex vector bundle, for some integer $\ell$, and use as a the twisting bundle $V = (TM \oplus \mathbb{R}^\ell)_C \otimes \text{Sym}^{k-2}H$ instead of $V = T^CM \otimes \text{Sym}^{k-2}H$. Then, observe that the additional term in the index formula is $\ell \text{ind}D^{k-2,0}$, which is 0 by Fact 4.

**Notice:** $S^4$ is weakly complex.
Inner symmetric spaces of compact type

Wolf spaces are all irreducible inner symmetric spaces of compact type, i.e. are of the form \( G/H \), where \( G \) is a compact simple Lie group and \( \text{rk} H = \text{rk} G \), i.e. a maximal torus of \( H \) is a maximal torus of \( G \).

Inner symmetric spaces of compact type are even-dimensional. The irreducible, simply-connected ones are:

1. The Wolf spaces
2. The irreducible Hermitian symmetric spaces = the adjoint orbits of compact simple Lie groups
3. The even-dimensional spheres \( S^{2\ell} \)
4. The real Grassmannians \( \tilde{G}r_{2p}(\mathbb{R}^{2p+n}) \)
5. The quaternionic Grassmannians \( Gr_k(\mathbb{H}^{k+\ell}) \)
6. The Cayley projective space \( F_4/\text{Spin}(9) \)
7. The two exceptional inner symmetric spaces \( E_7/(SU(8)/\mathbb{Z}_2) \) and \( E_8/(\text{Spin}(16)/\mathbb{Z}_2) \).
The techniques used above can be extended to a large class of inner symmetric spaces, in addition to Wolf spaces covered by Theorem 3, to get:

**Theorem 5** An irreducible, simply-connected, inner symmetric space of compact type of dimension $4k$ is weakly complex if and only if it is a sphere $S^{4k}$ or a Hermitian symmetric space (of even complex dimension).

**Remark 1.** Spheres of all dimensions are weakly complex, whereas Hermitian symmetric spaces of all (complex) dimensions are complex manifolds.

**Remark 2.** Our approach is ineffective for non-inner symmetric spaces of compact type, as the index of any homogeneous Dirac operator defined on such spaces is zero (R. Bott 1965).
Theorem 5 covers all inner symmetric spaces of compact type, except for:

1. The real Grassmannians of the form $\tilde{G}r_{2p}(\mathbb{R}^{2p+q})$, with $p$ and $q$ both odd

2. The exceptional inner symmetric space $E_7/(SU(8)/\mathbb{Z}_2)$, which is of dimension 70

Since the case of all even-dimensional real Grassmannians, except for $\tilde{G}r_4(\mathbb{R}^8)$, $\tilde{G}r_6(\mathbb{R}^{12})$ and $\tilde{G}r_4(\mathbb{R}^{10})$, was covered — by different methods — by P. Sankaran and Z.-Z. Tang op. cit., the only remaining open question about the existence issue of a (weak) complex structure on inner symmetric spaces of compact type is:

**Question.** Does there exists a (weak) complex structure on the exceptional inner symmetric space $E_7/(SU(8)/\mathbb{Z}_2)$?