Special quaternionic manifolds from variations of Hodge structures

A. Baarsma, J. Stienstra, T. van der Aalst, S. Vandoren (to appear)
A conversation...

- **Physicist**: “Quaternion-Kähler manifolds from T-dualizing vector multiplets into hypermultiplets in N=2 D=4 supergravity”
- **Mathematician**: “Excuse me?”
- **Physicist**: “Compactify type IIA strings on a CY$_3$ manifold, look at complex structure deformations, and add RR-fields + dilaton”
- **Mathematician**: “What are you talking about?”
Two years later ...

- **Mathematician:** “Aha! Start with variations of Hodge structures of CY type, construct bundle of Weil intermediate Jacobians, and add punctured discs. Is that it?”

- **Physicist:** “What are you talking about?”
Aim

- I will present a construction of quaternion-Kähler manifolds, discovered by physicists (Cecotti, Ferrara and Girardello, ‘91 – “the c-map”), but now translated into purely mathematical terms.

- [Related work by:
  M. Roček, C. Vafa, S.V. ‘06;
  N. Hitchin, ‘09;
  V. Cortés, ’98; V. Cortés, T Mohaupt, H. Xu, arXiv:1101.5103 (see Sunday-talk!)]

Quaternion-Kähler manifolds

- QK manifolds \((M,g,J)\) of dimension \(4n\) have holonomy \(\text{Sp}(n) \times \text{Sp}(1)\). They are Einstein. [Supersymmetry: \(R < 0!!\)]

- Quaternionic structure

\[
J^i J^j = - \delta^i_j \ 1 + \epsilon^{ijk} J^k
\]

- Not integrable: \(\text{Sp}(1)\) connection \(p\)

\[
\nabla J = p \times J \quad R(p) = gJ
\]
Content

- Variations of CY-Hodge structures
- Quaternionic vector bundles
- Hyperbolic spaces & Heisenberg groups
- Quaternion-Kähler manifolds
- Weil intermediate Jacobians
- The full picture
CY-Hodge structures

- Let $X$ be a compact Kähler manifold; (complexified) cohomology groups $H^k(X,\mathbb{C})$
- Hodge decomposition

\[ H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X) \]

- $k=3$ (CY$_3$, $\dim H^{3,0} = 1$):

\[ H_\mathcal{L} \equiv H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} = H_\circ \otimes \mathcal{L} \]
CY-Hodge structures

- Weil operator (complex structure on $H^3(X, \mathbb{R})$)

$$C |_{H^{p,q}} = i^{p-q}, \quad H^{p,q} = H^{q,p}$$

- Non-degenerate alternating form

$$Q(H^{p,q}, H^{p',q'}) = 0 \quad \text{if} \quad (p, q) \neq (q', p')$$

$$Q(a, Ca) > 0 \quad \forall a \in H^\circ \setminus \{0\}$$

$$Q(a, b) = \int_X a \wedge b$$
CY-Hodge structures

- Hodge filtration $F$:

$$F^q = \bigoplus_{p \geq q} H^{p,3-q}$$

$$H^{p,q} = F^p \cap F^q$$

- Instead of Hodge decomposition, we work with the filtration

$$0 = F^4 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = H^3(X,\mathcal{L})$$
Variation of Hodge structures

Family of compact Kähler manifolds (think of moduli space $M$ of complex structure deformations of $CY_3$)

$\downarrow$

Family of Hodge structures. Hodge bundle (Bryant & Griffiths, ‘83):

$$H^\cdot \rightarrow M$$

Variation of Hodge structure on base manifold $M$: $$(M, H^\cdot, \nabla_{GM}, F, Q)$$
Variation of Hodge structures

- Griffiths transversality:
  \[ \nabla_{GM} \bigotimes (F^p) \subseteq \Omega^1(F^{p-1}) \]

- Essentially, this means that
  \[ \phi : F^3 \otimes TM \rightarrow F^2 / F^3 \cong H^{2,1} \]
  \[ \Omega \otimes X \rightarrow \phi_{\Omega}(X) = [\nabla_X \Omega] \]

is a homomorphism. When M is such that this is an isomorphism, M is called (projective) special Kähler. [Cortés, Freed]
Special Kähler manifolds

- Hermitian form on $H_R$ (indefinite):

$$h = 2i Q(\cdot, \bar{\cdot})$$

- Restriction to $F^3 = H^{3,0}$ is positive definite and defines Kähler form and (Weil-Petersson) metric:

$$\Theta = -\partial\bar{\partial} \log(h(\Omega,\Omega))$$

$$K_{WP} = -\log(h(\Omega,\Omega))$$
Weil versus Griffiths

- Griffiths: \( h = 2iQ(\cdot, \overline{\cdot}) \) is positive definite on \( H^{3,0} + H^{1,2} \) and negative definite on \( H^{0,3} + H^{2,1} \). [ \( Q(a, Ca) > 0 \) ]

- Weil metric:

\[
h^W(\cdot, \cdot) = Q(-C, \cdot, \cdot) + iQ(\cdot, \cdot)
\]

is positive definite on entire \( H^0 \).

\( C: \) \(-i\ i\ -i\ i\)
 Quaternionic Vector Bundles

- Let $M$ be a special Kähler manifold. We can construct a vector bundle over $M$: 
  \[ Q \equiv \mathcal{L}_M \times \mathbb{R}^3 \]
- For each fibre above $t \in M, Q_t$ is isomorphic to the quaternionic algebra:
  \[ Q \otimes Q \rightarrow Q \]
  \[ (f, \Omega) \otimes (g, \Psi) \rightarrow (fg - h(\Omega, \Psi), f\Psi + g\bar{\Omega}) \]
Quaternionic Vector Bundles

- Define now the vector bundle
  
  \[ W \equiv F^3^* \otimes F^2 \oplus \bar{F}^2 \]

  \[ = \text{Hom}(F^3, F^2) \oplus \bar{F}^2 \]

  \[ = \text{Hom}(H^{3,0}, H^{3,0} \oplus H^{2,1}) \oplus H^{1,2} \oplus H^{0,3} \]

  \[ = \mathcal{L} \cdot 1 \oplus TM \oplus H^{1,2} \oplus H^{0,3} \]

- Right-action of the quaternions on \( W \) (fibre-wise):
  
  \[ W \otimes Q \rightarrow W \]
Quatertionic Vector Bundles

- Define metric

\[ g_W : W \otimes W \rightarrow \mathbb{O} \]

\[ g_W (w_1, w_2) = \text{Re}(h(\Omega_2, \Omega_1)h^W (\alpha_1, \alpha_2) + h^W (\beta_2, \beta_1)) \]

\[ w_{1,2} = \varphi_{\Omega_{1,2}} \otimes \alpha_{1,2} + \overline{\beta}_{1,2} \quad \varphi_{\Omega} \equiv h(\cdot, \Omega) \in F_3^* \]

- Metric compatible with quaternionic action:

\[ g_W (w_1 \cdot q, w_2 \cdot q) = \|q\|^2 g_W (w_1, w_2) \]
Further plan

- Consider the fibres $H_{\cdot,t}; t \in M$ of the Hodge bundle $H_{\cdot} \rightarrow M$.

- Statement 1: We can enlarge each fibre to become **complex hyperbolic space**.

- Statement 2: Bundle of hyperbolic spaces over $M$ is a quaternion-Kähler manifold.
Complex hyperbolic space

- Representation in terms of Siegel domain:

\[ S = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} \mid \Im(w) > \frac{1}{2} |z|^2\} \]

- Siegel domain associated to a Hodge structure:

\[ S^W = \{(a, \tau) \in H^\circ \times \mathbb{C} \mid \Im(\tau) > \frac{1}{2} h^W(a, a)\} \]
Complex hyperbolic space

- Bergmann metric $h_B$ on $S^W$ : Kähler, with Kähler potential
  \[ K_B(a, \tau) = -\log(2 \text{Im}(\tau) - h^W(a, a)) \]
- Siegel domain $S^W$ is diffeomorphic to the extended Heisenberg group
  \[ E_{\text{Heis}} = H_\circ \times \circ \times \circ + \]
- Group multiplication
  \[ (a, \sigma, \psi) \cdot (a', \sigma', \psi') = (a + \sqrt{\psi} a', \sigma + \psi \sigma' + \sqrt{\psi} Q(a, a'), \psi \psi') \]
Bundle of hyperbolic spaces

- Let \((M, \mathcal{H}, \nabla_{GM}, F, Q)\) be a special Kähler manifold, Hodge bundle \(\mathcal{H} \rightarrow M\).
- Define the product manifold

\[ M_{QK} = \mathcal{H} \times \circ \times \circ + \]

- Projection map \(\hat{\pi}: M_{QK} \rightarrow M\).
- Fibres

\[ E_{Heis_t} = \hat{\pi}^{-1}(t) = \mathcal{H}_{t} \times \circ \times \circ + \]
Metric

- We can construct a metric on $M_{QK}$:
  \[ g_{QK} = \hat{\pi}^* g_{WP} \oplus g_B \]

- Tangent bundle
  \[ TM_{QK} = \hat{\pi}^* TM \oplus \ker(d\hat{\pi}) \]
Quaternionic vector bundles

- Reminder: vector bundles over M:
\[ Q = \mathcal{E}_M \times F^3 \]
\[ W = F^{3*} \otimes F^2 \oplus \bar{F}^2 \]
\[ W \otimes Q \rightarrow W \]

- Pull back to vector bundles over \( M_{QK} \):
\[ \hat{Q} = \hat{\pi}^* Q \]
\[ \hat{W} = \hat{\pi}^* W \]

- One can construct an isomorphism
\[ \Phi : \hat{W} \rightarrow TM_{QK} \]
Pull back the metric $g_W$ on $W$ to $\hat{W}$

Theorem:

$$g_{QK}(\Phi w_1, \Phi w_2) = g_{\hat{W}}(w_1, w_2) \quad \forall w_{1,2} \in \hat{W}_t; W_t$$

Also after the pull back, $\hat{Q}$ acts linearly on $\hat{W}; TM_{QK}$, so $\hat{Q}$ provides subbundle of $\text{End}(TM_{QK})$, hence a quaternionic structure. One can show that $M_{QK}$ is QK!
Summary of further results

- We can construct bundle of intermediate Jacobians

\[ \text{Jac} = \frac{H^\circ}{H^\varsigma} \rightarrow M \]

- Combined with Weil complex structure and Weil metric, this becomes an abelian variety (torus).
Summary of further results

- One can take quotient of Siegel domain
  
  \[ E\text{Heis}(\circ) = H_\circ \times \circ \times \circ^+ \]
  
  \[ Heis(\mathcal{C}) \]

- Bundle of punctured disks over Jac over M

- For fixed \( \psi \in \circ^+ \), one gets circle bundle over the Weil intermediate Jacobian: contact geometry, Sasaki structure
The full picture (A. Baarsma)