# RELATIVE RATES OF GROWTH

It is often important to determine how fast functions $f(x)$ grow for very large values of x, and to compare the growth rate of various functions.

Ex 1: Any quadratic function grows faster than any linear function *eventually*. That is, even though for some values of x the quadratic function may have smaller magnitude and grow slower than the linear function, the quadratic growth will dominate the linear one *if x is large enough.* (Compare x and x^2, for example, as in Figure 1.)
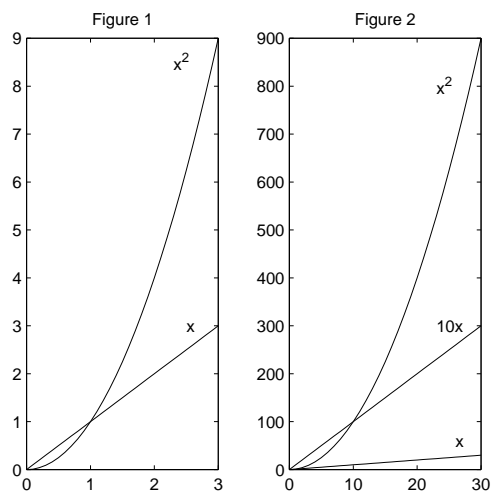
Ex 2: We know that while the values of one linear function may be larger than those of another, any two linear functions eventually grow slower than any quadratic function. (Figure 2 compares x, $10x$ and x^2, as an example.)

Figure 1

Figure 2

Ex 3: You may have noticed that exponential functions like 2^x and e^x seem to grow more rapidly as x gets large than polynomials and rational functions. Figure 3 compares e^x and 2^x with x^2. You can see the exponentials outgrowing x^2 as x increases. In fact, as $x \to \infty$, the functions 2^x and e^x grow faster than any power of x, even $x^{1,000,000}$.
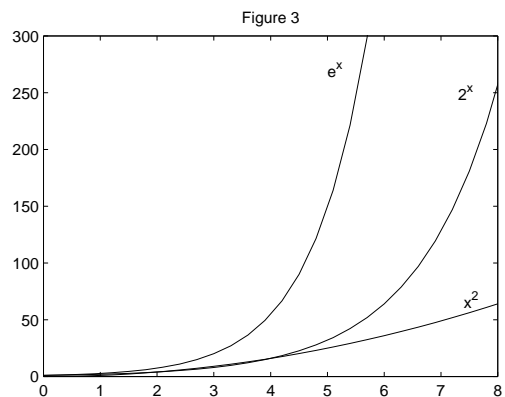
To get a feeling for how rapidly the values of $y = e^x$ grow with increasing x, think of graphing the function on a large blackboard, with the axes scaled in centimeters. At $x = 1$ cm, the graph is $e^1 \approx 3$ cm above the x-axis. At $x = 6$ cm, the graph is $e^6 \approx 403$ cm $\approx$ 4m high (probably higher than the

Figure 3

ceiling). At $x = 10$ cm, the graph is $e^{10} \approx 22,026$ cm $\approx$ 220 m high, higher than most buildings. At $x = 24$ cm, the graph is more than halfway to the moon, and at $x = 43$ cm, the graph is high enough to reach past the sun's closest stellar neighbor, the red dwarf star Proxima Centauri. Yet with $x = 43$ cm from the origin, the graph is still less than 2 feet to the right of the y-axis.

Here we want to compare, in particular, the growth rates of the new functions we have learned about (logarithms, exponentials), as well some of those we already know about (polynomials, square roots, other powers). The following definition precisely states what it means for one function to *grow faster than, grow slower than*, or *grow at the same rate as* another one, *eventually*, that is, if x is large enough. For present purposes, we restrict our attention to functions whose values eventually become and remain positive as $x \to \infty$.

Definition: Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. $f(x)$ grows faster than $g(x)$ as $x \to \infty$ if $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = \infty$.

2. $f(x)$ grows slower than $g(x)$ as $x \to \infty$ if $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = 0$.

3. $f(x)$ and $g(x)$ grow at the same rate as $x \to \infty$ if $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = L \neq 0$,
   where L is some finite number.

This definition implies that if f grows faster than g, then f will eventually be much larger than g. Similarly, if f grows slower than g, then f will eventually be much smaller than g.

In order to compute the limits involved we often use L'Hôpital's rule.

The notion of relative rates of growth will be very useful to us later on in this semester, when we talk about integrals over infinite domains and when we talk about series. In the problems below you will establish, among others, that:

Any two polynomial functions of equal degree grow at the same rate.

x^a grows slower than x^b if $a < b$.

Logarithms grow slower than polynomials which grow slower than (growing) exponentials.

Note: Convince yourself that 2^x is a growing exponential, while $\left(\frac{1}{2}\right)^x$ is decaying as $x \to \infty$.

Homework Problems:

1. (a) Show that x^2 grows faster than x as $x \to \infty$.
   (b) Show that x and $10x$ grow at the same rate as $x \to \infty$.

2. (a) Show that e^x grows faster than x^2 as $x \to \infty$.
   (b) Show that e^x grows faster than 2^x as $x \to \infty$.

3. (a) Show that $\ln x$ grows slower than x as $x \to \infty$.
   (b) Show that $\ln x$ grows at the same rate as $\ln(x^2)$ as $x \to \infty$.

4. Show that any quadratic function $f(x) = a_2 x^2 + a_1 x + a_0$ grows faster than any linear function $g(x) = b_1 x + b_0$, where $a_2, b_1 > 0$, as $x \to \infty$.

5. Show that any two linear functions, $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x$, $a_1, b_1 > 0$, grow at the same rate (namely linearly) as $x \to \infty$. *Similarly, one can show that any two polynomial functions of equal degree grow at the same rate.*

6. (a) Show that $\log_a(x)$ and $\log_b(x)$ grow at the same rate as $x \to \infty$.

   (b) Show that a^x grows slower than b^x as $x \to \infty$, if $a < b$.
   For example, 2^x grows slower than e^x.

   (c) Show that x^a grows slower than x^b if $a < b$.
   For example, $\sqrt{x}$ grows slower than x which grows slower than x^2.

7. Show that $\ln(x)$ grows at the same rate as $\ln(x^3 - 3x + 1)$.

8.* Let n be a positive integer.
   (a) Show that $\ln(x)$ grows slower than x^n as $x \to \infty$.
   (a) Show that x^n grows slower than e^x as $x \to \infty$.

9. Which of the following functions grow faster than x^2? Which grow at the same rate as x^2? Which grow slower?

(a)	$x^2 + 4x$	(h)	2^x
(b)	$\sqrt{x}$	(i)	$10x^2$
(c)	$x^2 + \sqrt{x}$	(j)	$(1.1)^x$
(d)	$x^3 - x^2$	(k)	$(0.9)^x$
(e)	$\sqrt{x^4 - x^3}$	(l)	$\log_{10} x^2$
(f)	$x \ln x$	(m)	x^x
(g)	$x^3 e^{-x}$		

10.* Which of the following functions grow faster than $\ln x$? Which grow at the same rate as $\ln x$? Which grow slower?

(a)	$\log_3 x$	(g)	$5 \ln x$
(b)	$\ln \sqrt{x}$	(h)	e^x
(c)	x	(i)	$\log_2(x^2)$
(d)	$1/x$	(j)	$\log_{10}(10x)$
(e)	$\ln(2x)$	(k)	$\ln(\ln x)$
(f)	$\sqrt{x}$	(l)	$10 \ln x + x$

11.* Order the following functions from slowest growing to fastest growing, as $x \to \infty$. Group functions that grow at the same rate together.

(a)	e^x	(h)	x^2
(b)	x^x	(i)	$\sqrt{x}$
(c)	$\ln(x)^x$	(j)	$(\ln 2)^x$
(d)	$e^{x/2}$	(k)	$\ln(x^2)$
(e)	2^x	(l)	e^{x^2}
(f)	$(0.9)^x$	(m)	x
(g)	$1/x^2$	(n)	$1/x$